



FHSST Authors

**The Free High School Science Texts:
Textbooks for High School Students
Studying the Sciences
Mathematics
Grades 10 - 12**

**Version 0
September 17, 2008**

Copyright 2007 "Free High School Science Texts"

Permission **is** granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".



STOP!!!!

Did you notice the FREEDOMS we've granted you?

Our copyright license is **different!** It grants freedoms rather than just imposing restrictions like all those other textbooks you probably own or use.

- We know people copy textbooks illegally but we would LOVE it if you copied our's - go ahead copy to your hearts content, **legally!**
- Publishers revenue is generated by controlling the market, we don't want any money, go ahead, distribute our books far and wide - we DARE you!
- Ever wanted to change your textbook? Of course you have! Go ahead change ours, make your own version, get your friends together, rip it apart and put it back together the way you like it. That's what we really want!
- Copy, modify, adapt, enhance, share, critique, adore, and contextualise. Do it all, do it with your colleagues, your friends or alone but get involved! Together we can overcome the challenges our complex and diverse country presents.
- So what is the catch? The only thing you can't do is take this book, make a few changes and then tell others that they can't do the same with your changes. It's share and share-alike and we know you'll agree that is only fair.
- These books were written by volunteers who want to help support education, who want the facts to be freely available for teachers to copy, adapt and re-use. Thousands of hours went into making them and they are a gift to everyone in the education community.

FHSST Core Team

Mark Horner ; Samuel Halliday ; Sarah Blyth ; Rory Adams ; Spencer Wheaton

FHSST Editors

Jaynie Padayachee ; Joanne Boule ; Diana Mulcahy ; Annette Nell ; René Toerien ; Donovan Whitfield

FHSST Contributors

Rory Adams ; Prashant Arora ; Richard Baxter ; Dr. Sarah Blyth ; Sebastian Bodenstein ; Graeme Broster ; Richard Case ; Brett Cocks ; Tim Crombie ; Dr. Anne Dabrowski ; Laura Daniels ; Sean Dobbs ; Fernando Durrell ; Dr. Dan Dwyer ; Frans van Eeden ; Giovanni Franzoni ; Ingrid von Glehn ; Tamara von Glehn ; Lindsay Glesener ; Dr. Vanessa Godfrey ; Dr. Johan Gonzalez ; Hemant Gopal ; Umeshree Govender ; Heather Gray ; Lynn Greeff ; Dr. Tom Gutierrez ; Brooke Haag ; Kate Hadley ; Dr. Sam Halliday ; Asheena Hanuman ; Neil Hart ; Nicholas Hatcher ; Dr. Mark Horner ; Mfandaizda Hove ; Robert Hovden ; Jennifer Hsieh ; Clare Johnson ; Luke Jordan ; Tana Joseph ; Dr. Jennifer Klay ; Lara Kruger ; Sihle Kubheka ; Andrew Kubik ; Dr. Marco van Leeuwen ; Dr. Anton Machacek ; Dr. Komal Maheshwari ; Kosma von Maltitz ; Nicole Masureik ; John Mathew ; JoEllen McBride ; Nikolai Meures ; Riana Meyer ; Jenny Miller ; Abdul Mirza ; Asogan Moodaly ; Jothi Moodley ; Nolene Naidu ; Tyrone Negus ; Thomas O'Donnell ; Dr. Markus Oldenburg ; Dr. Jaynie Padayachee ; Nicolette Pekeur ; Sirika Pillay ; Jacques Plaut ; Andrea Prinsloo ; Joseph Raimondo ; Sanya Rajani ; Prof. Sergey Rakityansky ; Alastair Ramlakan ; Razvan Remsing ; Max Richter ; Sean Riddle ; Evan Robinson ; Dr. Andrew Rose ; Bianca Ruddy ; Katie Russell ; Duncan Scott ; Helen Seals ; Ian Sherratt ; Roger Sieloff ; Bradley Smith ; Greg Solomon ; Mike Stringer ; Shen Tian ; Robert Torregrosa ; Jimmy Tseng ; Helen Waugh ; Dr. Dawn Webber ; Michelle Wen ; Dr. Alexander Wetzler ; Dr. Spencer Wheaton ; Vivian White ; Dr. Gerald Wigger ; Harry Wiggins ; Wendy Williams ; Julie Wilson ; Andrew Wood ; Emma Wormauld ; Sahal Yacoob ; Jean Youssef

Contributors and editors have made a sincere effort to produce an accurate and useful resource. Should you have suggestions, find mistakes or be prepared to donate material for inclusion, please don't hesitate to contact us. We intend to work with all who are willing to help make this a continuously evolving resource!

www.fhsst.org

Contents

I	Basics	1
1	Introduction to Book	3
1.1	The Language of Mathematics	3
II	Grade 10	5
2	Review of Past Work	7
2.1	Introduction	7
2.2	What is a number?	7
2.3	Sets	7
2.4	Letters and Arithmetic	8
2.5	Addition and Subtraction	9
2.6	Multiplication and Division	9
2.7	Brackets	9
2.8	Negative Numbers	10
2.8.1	What is a negative number?	10
2.8.2	Working with Negative Numbers	11
2.8.3	Living Without the Number Line	12
2.9	Rearranging Equations	13
2.10	Fractions and Decimal Numbers	15
2.11	Scientific Notation	16
2.12	Real Numbers	16
2.12.1	Natural Numbers	17
2.12.2	Integers	17
2.12.3	Rational Numbers	17
2.12.4	Irrational Numbers	19
2.13	Mathematical Symbols	20
2.14	Infinity	20
2.15	End of Chapter Exercises	21
3	Rational Numbers - Grade 10	23
3.1	Introduction	23
3.2	The Big Picture of Numbers	23
3.3	Definition	23

3.4	Forms of Rational Numbers	24
3.5	Converting Terminating Decimals into Rational Numbers	25
3.6	Converting Repeating Decimals into Rational Numbers	25
3.7	Summary	26
3.8	End of Chapter Exercises	27
4	Exponentials - Grade 10	29
4.1	Introduction	29
4.2	Definition	29
4.3	Laws of Exponents	30
4.3.1	Exponential Law 1: $a^0 = 1$	30
4.3.2	Exponential Law 2: $a^m \times a^n = a^{m+n}$	30
4.3.3	Exponential Law 3: $a^{-n} = \frac{1}{a^n}, a \neq 0$	31
4.3.4	Exponential Law 4: $a^m \div a^n = a^{m-n}$	32
4.3.5	Exponential Law 5: $(ab)^n = a^n b^n$	32
4.3.6	Exponential Law 6: $(a^m)^n = a^{mn}$	33
4.4	End of Chapter Exercises	34
5	Estimating Surds - Grade 10	37
5.1	Introduction	37
5.2	Drawing Surds on the Number Line (Optional)	38
5.3	End of Chapter Exercises	39
6	Irrational Numbers and Rounding Off - Grade 10	41
6.1	Introduction	41
6.2	Irrational Numbers	41
6.3	Rounding Off	42
6.4	End of Chapter Exercises	43
7	Number Patterns - Grade 10	45
7.1	Common Number Patterns	45
7.1.1	Special Sequences	46
7.2	Make your own Number Patterns	46
7.3	Notation	47
7.3.1	Patterns and Conjecture	49
7.4	Exercises	50
8	Finance - Grade 10	53
8.1	Introduction	53
8.2	Foreign Exchange Rates	53
8.2.1	How much is R1 really worth?	53
8.2.2	Cross Currency Exchange Rates	56
8.2.3	Enrichment: Fluctuating exchange rates	57
8.3	Being Interested in Interest	58

8.4	Simple Interest	59
8.4.1	Other Applications of the Simple Interest Formula	61
8.5	Compound Interest	63
8.5.1	Fractions add up to the Whole	65
8.5.2	The Power of Compound Interest	65
8.5.3	Other Applications of Compound Growth	67
8.6	Summary	68
8.6.1	Definitions	68
8.6.2	Equations	68
8.7	End of Chapter Exercises	69
9	Products and Factors - Grade 10	71
9.1	Introduction	71
9.2	Recap of Earlier Work	71
9.2.1	Parts of an Expression	71
9.2.2	Product of Two Binomials	71
9.2.3	Factorisation	72
9.3	More Products	74
9.4	Factorising a Quadratic	76
9.5	Factorisation by Grouping	79
9.6	Simplification of Fractions	80
9.7	End of Chapter Exercises	82
10	Equations and Inequalities - Grade 10	83
10.1	Strategy for Solving Equations	83
10.2	Solving Linear Equations	84
10.3	Solving Quadratic Equations	89
10.4	Exponential Equations of the form $ka^{(x+p)} = m$	93
10.4.1	Algebraic Solution	93
10.5	Linear Inequalities	96
10.6	Linear Simultaneous Equations	99
10.6.1	Finding solutions	99
10.6.2	Graphical Solution	99
10.6.3	Solution by Substitution	101
10.7	Mathematical Models	103
10.7.1	Introduction	103
10.7.2	Problem Solving Strategy	104
10.7.3	Application of Mathematical Modelling	104
10.7.4	End of Chapter Exercises	106
10.8	Introduction to Functions and Graphs	107
10.9	Functions and Graphs in the Real-World	107
10.10	Recap	107

10.10.1 Variables and Constants	107
10.10.2 Relations and Functions	108
10.10.3 The Cartesian Plane	108
10.10.4 Drawing Graphs	109
10.10.5 Notation used for Functions	110
10.11 Characteristics of Functions - All Grades	112
10.11.1 Dependent and Independent Variables	112
10.11.2 Domain and Range	113
10.11.3 Intercepts with the Axes	113
10.11.4 Turning Points	114
10.11.5 Asymptotes	114
10.11.6 Lines of Symmetry	114
10.11.7 Intervals on which the Function Increases/Decreases	114
10.11.8 Discrete or Continuous Nature of the Graph	114
10.12 Graphs of Functions	116
10.12.1 Functions of the form $y = ax + q$	116
10.12.2 Functions of the Form $y = ax^2 + q$	120
10.12.3 Functions of the Form $y = \frac{a}{x} + q$	125
10.12.4 Functions of the Form $y = ab^{(x)} + q$	129
10.13 End of Chapter Exercises	133
11 Average Gradient - Grade 10 Extension	135
11.1 Introduction	135
11.2 Straight-Line Functions	135
11.3 Parabolic Functions	136
11.4 End of Chapter Exercises	138
12 Geometry Basics	139
12.1 Introduction	139
12.2 Points and Lines	139
12.3 Angles	140
12.3.1 Measuring angles	141
12.3.2 Special Angles	141
12.3.3 Special Angle Pairs	143
12.3.4 Parallel Lines intersected by Transversal Lines	143
12.4 Polygons	147
12.4.1 Triangles	147
12.4.2 Quadrilaterals	152
12.4.3 Other polygons	155
12.4.4 Extra	156
12.5 Exercises	157
12.5.1 Challenge Problem	159

13 Geometry - Grade 10	161
13.1 Introduction	161
13.2 Right Prisms and Cylinders	161
13.2.1 Surface Area	162
13.2.2 Volume	164
13.3 Polygons	167
13.3.1 Similarity of Polygons	167
13.4 Co-ordinate Geometry	171
13.4.1 Introduction	171
13.4.2 Distance between Two Points	172
13.4.3 Calculation of the Gradient of a Line	173
13.4.4 Midpoint of a Line	174
13.5 Transformations	177
13.5.1 Translation of a Point	177
13.5.2 Reflection of a Point	179
13.6 End of Chapter Exercises	185
14 Trigonometry - Grade 10	189
14.1 Introduction	189
14.2 Where Trigonometry is Used	190
14.3 Similarity of Triangles	190
14.4 Definition of the Trigonometric Functions	191
14.5 Simple Applications of Trigonometric Functions	195
14.5.1 Height and Depth	195
14.5.2 Maps and Plans	197
14.6 Graphs of Trigonometric Functions	199
14.6.1 Graph of $\sin \theta$	199
14.6.2 Functions of the form $y = a \sin(x) + q$	200
14.6.3 Graph of $\cos \theta$	202
14.6.4 Functions of the form $y = a \cos(x) + q$	202
14.6.5 Comparison of Graphs of $\sin \theta$ and $\cos \theta$	204
14.6.6 Graph of $\tan \theta$	204
14.6.7 Functions of the form $y = a \tan(x) + q$	205
14.7 End of Chapter Exercises	208
15 Statistics - Grade 10	211
15.1 Introduction	211
15.2 Recap of Earlier Work	211
15.2.1 Data and Data Collection	211
15.2.2 Methods of Data Collection	212
15.2.3 Samples and Populations	213
15.3 Example Data Sets	213

15.3.1 Data Set 1: Tossing a Coin	213
15.3.2 Data Set 2: Casting a die	213
15.3.3 Data Set 3: Mass of a Loaf of Bread	214
15.3.4 Data Set 4: Global Temperature	214
15.3.5 Data Set 5: Price of Petrol	215
15.4 Grouping Data	215
15.4.1 Exercises - Grouping Data	216
15.5 Graphical Representation of Data	217
15.5.1 Bar and Compound Bar Graphs	217
15.5.2 Histograms and Frequency Polygons	217
15.5.3 Pie Charts	219
15.5.4 Line and Broken Line Graphs	220
15.5.5 Exercises - Graphical Representation of Data	221
15.6 Summarising Data	222
15.6.1 Measures of Central Tendency	222
15.6.2 Measures of Dispersion	225
15.6.3 Exercises - Summarising Data	228
15.7 Misuse of Statistics	229
15.7.1 Exercises - Misuse of Statistics	230
15.8 Summary of Definitions	232
15.9 Exercises	232
16 Probability - Grade 10	235
16.1 Introduction	235
16.2 Random Experiments	235
16.2.1 Sample Space of a Random Experiment	235
16.3 Probability Models	238
16.3.1 Classical Theory of Probability	239
16.4 Relative Frequency vs. Probability	240
16.5 Project Idea	242
16.6 Probability Identities	242
16.7 Mutually Exclusive Events	243
16.8 Complementary Events	244
16.9 End of Chapter Exercises	246
III Grade 11	249
17 Exponents - Grade 11	251
17.1 Introduction	251
17.2 Laws of Exponents	251
17.2.1 Exponential Law 7: $a^{\frac{m}{n}} = \sqrt[n]{a^m}$	251
17.3 Exponentials in the Real-World	253
17.4 End of chapter Exercises	254

18 Surds - Grade 11	255
18.1 Surd Calculations	255
18.1.1 Surd Law 1: $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$	255
18.1.2 Surd Law 2: $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	255
18.1.3 Surd Law 3: $\sqrt[n]{a^m} = a^{\frac{m}{n}}$	256
18.1.4 Like and Unlike Surds	256
18.1.5 Simplest Surd form	257
18.1.6 Rationalising Denominators	258
18.2 End of Chapter Exercises	259
19 Error Margins - Grade 11	261
20 Quadratic Sequences - Grade 11	265
20.1 Introduction	265
20.2 What is a <i>quadratic sequence</i> ?	265
20.3 End of chapter Exercises	269
21 Finance - Grade 11	271
21.1 Introduction	271
21.2 Depreciation	271
21.3 Simple Depreciation (it really is simple!)	271
21.4 Compound Depreciation	274
21.5 Present Values or Future Values of an Investment or Loan	276
21.5.1 Now or Later	276
21.6 Finding i	278
21.7 Finding n - Trial and Error	279
21.8 Nominal and Effective Interest Rates	280
21.8.1 The General Formula	281
21.8.2 De-coding the Terminology	282
21.9 Formulae Sheet	284
21.9.1 Definitions	284
21.9.2 Equations	285
21.10 End of Chapter Exercises	285
22 Solving Quadratic Equations - Grade 11	287
22.1 Introduction	287
22.2 Solution by Factorisation	287
22.3 Solution by Completing the Square	290
22.4 Solution by the Quadratic Formula	293
22.5 Finding an equation when you know its roots	296
22.6 End of Chapter Exercises	299

23 Solving Quadratic Inequalities - Grade 11	301
23.1 Introduction	301
23.2 Quadratic Inequalities	301
23.3 End of Chapter Exercises	304
24 Solving Simultaneous Equations - Grade 11	307
24.1 Graphical Solution	307
24.2 Algebraic Solution	309
25 Mathematical Models - Grade 11	313
25.1 Real-World Applications: Mathematical Models	313
25.2 End of Chapter Exercises	317
26 Quadratic Functions and Graphs - Grade 11	321
26.1 Introduction	321
26.2 Functions of the Form $y = a(x + p)^2 + q$	321
26.2.1 Domain and Range	322
26.2.2 Intercepts	323
26.2.3 Turning Points	324
26.2.4 Axes of Symmetry	325
26.2.5 Sketching Graphs of the Form $f(x) = a(x + p)^2 + q$	325
26.2.6 Writing an equation of a shifted parabola	327
26.3 End of Chapter Exercises	327
27 Hyperbolic Functions and Graphs - Grade 11	329
27.1 Introduction	329
27.2 Functions of the Form $y = \frac{a}{x+p} + q$	329
27.2.1 Domain and Range	330
27.2.2 Intercepts	331
27.2.3 Asymptotes	332
27.2.4 Sketching Graphs of the Form $f(x) = \frac{a}{x+p} + q$	333
27.3 End of Chapter Exercises	333
28 Exponential Functions and Graphs - Grade 11	335
28.1 Introduction	335
28.2 Functions of the Form $y = ab^{(x+p)} + q$	335
28.2.1 Domain and Range	336
28.2.2 Intercepts	337
28.2.3 Asymptotes	338
28.2.4 Sketching Graphs of the Form $f(x) = ab^{(x+p)} + q$	338
28.3 End of Chapter Exercises	339
29 Gradient at a Point - Grade 11	341
29.1 Introduction	341
29.2 Average Gradient	341
29.3 End of Chapter Exercises	344

30 Linear Programming - Grade 11	345
30.1 Introduction	345
30.2 Terminology	345
30.2.1 Decision Variables	345
30.2.2 Objective Function	345
30.2.3 Constraints	346
30.2.4 Feasible Region and Points	346
30.2.5 The Solution	346
30.3 Example of a Problem	347
30.4 Method of Linear Programming	347
30.5 Skills you will need	347
30.5.1 Writing Constraint Equations	347
30.5.2 Writing the Objective Function	348
30.5.3 Solving the Problem	350
30.6 End of Chapter Exercises	352
31 Geometry - Grade 11	357
31.1 Introduction	357
31.2 Right Pyramids, Right Cones and Spheres	357
31.3 Similarity of Polygons	360
31.4 Triangle Geometry	361
31.4.1 Proportion	361
31.5 Co-ordinate Geometry	368
31.5.1 Equation of a Line between Two Points	368
31.5.2 Equation of a Line through One Point and Parallel or Perpendicular to Another Line	371
31.5.3 Inclination of a Line	371
31.6 Transformations	373
31.6.1 Rotation of a Point	373
31.6.2 Enlargement of a Polygon 1	376
32 Trigonometry - Grade 11	381
32.1 History of Trigonometry	381
32.2 Graphs of Trigonometric Functions	381
32.2.1 Functions of the form $y = \sin(k\theta)$	381
32.2.2 Functions of the form $y = \cos(k\theta)$	383
32.2.3 Functions of the form $y = \tan(k\theta)$	384
32.2.4 Functions of the form $y = \sin(\theta + p)$	385
32.2.5 Functions of the form $y = \cos(\theta + p)$	386
32.2.6 Functions of the form $y = \tan(\theta + p)$	387
32.3 Trigonometric Identities	389
32.3.1 Deriving Values of Trigonometric Functions for 30° , 45° and 60°	389
32.3.2 Alternate Definition for $\tan \theta$	391

32.3.3	A Trigonometric Identity	392
32.3.4	Reduction Formula	394
32.4	Solving Trigonometric Equations	399
32.4.1	Graphical Solution	399
32.4.2	Algebraic Solution	401
32.4.3	Solution using CAST diagrams	403
32.4.4	General Solution Using Periodicity	405
32.4.5	Linear Trigonometric Equations	406
32.4.6	Quadratic and Higher Order Trigonometric Equations	406
32.4.7	More Complex Trigonometric Equations	407
32.5	Sine and Cosine Identities	409
32.5.1	The Sine Rule	409
32.5.2	The Cosine Rule	412
32.5.3	The Area Rule	414
32.6	Exercises	416
33	Statistics - Grade 11	419
33.1	Introduction	419
33.2	Standard Deviation and Variance	419
33.2.1	Variance	419
33.2.2	Standard Deviation	421
33.2.3	Interpretation and Application	423
33.2.4	Relationship between Standard Deviation and the Mean	424
33.3	Graphical Representation of Measures of Central Tendency and Dispersion	424
33.3.1	Five Number Summary	424
33.3.2	Box and Whisker Diagrams	425
33.3.3	Cumulative Histograms	426
33.4	Distribution of Data	428
33.4.1	Symmetric and Skewed Data	428
33.4.2	Relationship of the Mean, Median, and Mode	428
33.5	Scatter Plots	429
33.6	Misuse of Statistics	432
33.7	End of Chapter Exercises	435
34	Independent and Dependent Events - Grade 11	437
34.1	Introduction	437
34.2	Definitions	437
34.2.1	Identification of Independent and Dependent Events	438
34.3	End of Chapter Exercises	441
IV	Grade 12	443
35	Logarithms - Grade 12	445
35.1	Definition of Logarithms	445

35.2	Logarithm Bases	446
35.3	Laws of Logarithms	447
35.4	Logarithm Law 1: $\log_a 1 = 0$	447
35.5	Logarithm Law 2: $\log_a(a) = 1$	448
35.6	Logarithm Law 3: $\log_a(x \cdot y) = \log_a(x) + \log_a(y)$	448
35.7	Logarithm Law 4: $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$	449
35.8	Logarithm Law 5: $\log_a(x^b) = b \log_a(x)$	450
35.9	Logarithm Law 6: $\log_a(\sqrt[b]{x}) = \frac{\log_a(x)}{b}$	450
35.10	Solving simple log equations	452
35.10.1	Exercises	454
35.11	Logarithmic applications in the Real World	454
35.11.1	Exercises	455
35.12	End of Chapter Exercises	455
36	Sequences and Series - Grade 12	457
36.1	Introduction	457
36.2	Arithmetic Sequences	457
36.2.1	General Equation for the n^{th} -term of an Arithmetic Sequence	458
36.3	Geometric Sequences	459
36.3.1	Example - A Flu Epidemic	459
36.3.2	General Equation for the n^{th} -term of a Geometric Sequence	461
36.3.3	Exercises	461
36.4	Recursive Formulae for Sequences	462
36.5	Series	463
36.5.1	Some Basics	463
36.5.2	Sigma Notation	463
36.6	Finite Arithmetic Series	465
36.6.1	General Formula for a Finite Arithmetic Series	466
36.6.2	Exercises	467
36.7	Finite Squared Series	468
36.8	Finite Geometric Series	469
36.8.1	Exercises	470
36.9	Infinite Series	471
36.9.1	Infinite Geometric Series	471
36.9.2	Exercises	472
36.10	End of Chapter Exercises	472
37	Finance - Grade 12	477
37.1	Introduction	477
37.2	Finding the Length of the Investment or Loan	477
37.3	A Series of Payments	478
37.3.1	Sequences and Series	479

37.3.2 Present Values of a series of Payments	479
37.3.3 Future Value of a series of Payments	484
37.3.4 Exercises - Present and Future Values	485
37.4 Investments and Loans	485
37.4.1 Loan Schedules	485
37.4.2 Exercises - Investments and Loans	489
37.4.3 Calculating Capital Outstanding	489
37.5 Formulae Sheet	489
37.5.1 Definitions	490
37.5.2 Equations	490
37.6 End of Chapter Exercises	490
38 Factorising Cubic Polynomials - Grade 12	493
38.1 Introduction	493
38.2 The Factor Theorem	493
38.3 Factorisation of Cubic Polynomials	494
38.4 Exercises - Using Factor Theorem	496
38.5 Solving Cubic Equations	496
38.5.1 Exercises - Solving of Cubic Equations	498
38.6 End of Chapter Exercises	498
39 Functions and Graphs - Grade 12	501
39.1 Introduction	501
39.2 Definition of a Function	501
39.2.1 Exercises	501
39.3 Notation used for Functions	502
39.4 Graphs of Inverse Functions	502
39.4.1 Inverse Function of $y = ax + q$	503
39.4.2 Exercises	504
39.4.3 Inverse Function of $y = ax^2$	504
39.4.4 Exercises	504
39.4.5 Inverse Function of $y = a^x$	506
39.4.6 Exercises	506
39.5 End of Chapter Exercises	507
40 Differential Calculus - Grade 12	509
40.1 Why do I have to learn this stuff?	509
40.2 Limits	510
40.2.1 A Tale of Achilles and the Tortoise	510
40.2.2 Sequences, Series and Functions	511
40.2.3 Limits	512
40.2.4 Average Gradient and Gradient at a Point	516
40.3 Differentiation from First Principles	519

40.4	Rules of Differentiation	521
40.4.1	Summary of Differentiation Rules	522
40.5	Applying Differentiation to Draw Graphs	523
40.5.1	Finding Equations of Tangents to Curves	523
40.5.2	Curve Sketching	524
40.5.3	Local minimum, Local maximum and Point of Inflexion	529
40.6	Using Differential Calculus to Solve Problems	530
40.6.1	Rate of Change problems	534
40.7	End of Chapter Exercises	535
41	Linear Programming - Grade 12	539
41.1	Introduction	539
41.2	Terminology	539
41.2.1	Feasible Region and Points	539
41.3	Linear Programming and the Feasible Region	540
41.4	End of Chapter Exercises	546
42	Geometry - Grade 12	549
42.1	Introduction	549
42.2	Circle Geometry	549
42.2.1	Terminology	549
42.2.2	Axioms	550
42.2.3	Theorems of the Geometry of Circles	550
42.3	Co-ordinate Geometry	566
42.3.1	Equation of a Circle	566
42.3.2	Equation of a Tangent to a Circle at a Point on the Circle	569
42.4	Transformations	571
42.4.1	Rotation of a Point about an angle θ	571
42.4.2	Characteristics of Transformations	573
42.4.3	Characteristics of Transformations	573
42.5	Exercises	574
43	Trigonometry - Grade 12	577
43.1	Compound Angle Identities	577
43.1.1	Derivation of $\sin(\alpha + \beta)$	577
43.1.2	Derivation of $\sin(\alpha - \beta)$	578
43.1.3	Derivation of $\cos(\alpha + \beta)$	578
43.1.4	Derivation of $\cos(\alpha - \beta)$	579
43.1.5	Derivation of $\sin 2\alpha$	579
43.1.6	Derivation of $\cos 2\alpha$	579
43.1.7	Problem-solving Strategy for Identities	580
43.2	Applications of Trigonometric Functions	582
43.2.1	Problems in Two Dimensions	582

43.2.2 Problems in 3 dimensions	584
43.3 Other Geometries	586
43.3.1 Taxicab Geometry	586
43.3.2 Manhattan distance	586
43.3.3 Spherical Geometry	587
43.3.4 Fractal Geometry	588
43.4 End of Chapter Exercises	589
44 Statistics - Grade 12	591
44.1 Introduction	591
44.2 A Normal Distribution	591
44.3 Extracting a Sample Population	593
44.4 Function Fitting and Regression Analysis	594
44.4.1 The Method of Least Squares	596
44.4.2 Using a calculator	597
44.4.3 Correlation coefficients	599
44.5 Exercises	600
45 Combinations and Permutations - Grade 12	603
45.1 Introduction	603
45.2 Counting	603
45.2.1 Making a List	603
45.2.2 Tree Diagrams	604
45.3 Notation	604
45.3.1 The Factorial Notation	604
45.4 The Fundamental Counting Principle	604
45.5 Combinations	605
45.5.1 Counting Combinations	605
45.5.2 Combinatorics and Probability	606
45.6 Permutations	606
45.6.1 Counting Permutations	607
45.7 Applications	608
45.8 Exercises	610
V Exercises	613
46 General Exercises	615
47 Exercises - Not covered in Syllabus	617
A GNU Free Documentation License	619

Chapter 4

Exponentials - Grade 10

4.1 Introduction

In this chapter, you will learn about the short-cuts to writing $2 \times 2 \times 2 \times 2$. This is known as writing a number in *exponential notation*.

4.2 Definition

Exponential notation is a short way of writing the same number multiplied by itself many times. For example, instead of $5 \times 5 \times 5$, we write 5^3 to show that the number 5 is multiplied by itself 3 times and we say "5 to the power of 3". Likewise 5^2 is 5×5 and 3^5 is $3 \times 3 \times 3 \times 3 \times 3$. We will now have a closer look at writing numbers using exponential notation.



Definition: Exponential Notation

Exponential notation means a number written like

$$a^n$$

when n is an integer and a can be any real number. a is called the *base* and n is called the *exponent*.

The n th power of a is defined as:

$$a^n = 1 \times a \times a \times \dots \times a \quad (n \text{ times}) \quad (4.1)$$

with a multiplied by itself n times.

We can also define what it means if we have a negative index, $-n$. Then,

$$a^{-n} = 1 \div a \div a \div \dots \div a \quad (n \text{ times}) \quad (4.2)$$



Important: Exponentials

If n is an even integer, then a^n will always be positive for any non-zero real number a . For example, although -2 is negative, $(-2)^2 = 1 \times -2 \times -2 = 4$ is positive and so is $(-2)^{-2} = 1 \div -2 \div -2 = \frac{1}{4}$.

4.3 Laws of Exponents

There are several laws we can use to make working with exponential numbers easier. Some of these laws might have been seen in earlier grades, but we will list all the laws here for easy reference, but we will explain each law in detail, so that you can understand them, and not only remember them.

$$a^0 = 1 \quad (4.3)$$

$$a^m \times a^n = a^{m+n} \quad (4.4)$$

$$a^{-n} = \frac{1}{a^n} \quad (4.5)$$

$$a^m \div a^n = a^{m-n} \quad (4.6)$$

$$(ab)^n = a^n b^n \quad (4.7)$$

$$(a^m)^n = a^{mn} \quad (4.8)$$

4.3.1 Exponential Law 1: $a^0 = 1$

Our definition of exponential notation shows that

$$a^0 = 1, (a \neq 0) \quad (4.9)$$

For example, $x^0 = 1$ and $(1\,000\,000)^0 = 1$.



Exercise: Application using Exponential Law 1: $a^0 = 1, (a \neq 0)$

1. $16^0 = 1$
 2. $16a^0 = 16$
 3. $(16 + a)^0 = 1$
 4. $(-16)^0 = 1$
 5. $-16^0 = -1$
-

4.3.2 Exponential Law 2: $a^m \times a^n = a^{m+n}$

Our definition of exponential notation shows that

$$\begin{aligned} a^m \times a^n &= 1 \times a \times \dots \times a && (m \text{ times}) \\ &\times 1 \times a \times \dots \times a && (n \text{ times}) \\ &= 1 \times a \times \dots \times a && (m + n \text{ times}) \\ &= a^{m+n} \end{aligned} \quad (4.10)$$

For example,

$$\begin{aligned} 2^7 \times 2^3 &= (2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2) \times (2 \times 2 \times 2) \\ &= 2^{10} \\ &= 2^{7+3} \end{aligned}$$



This simple law is the reason why exponentials were originally invented. In the days before calculators, all multiplication had to be done by hand with a pencil and a pad of paper. Multiplication takes a very long time to do and is very tedious. Adding numbers however, is very easy and quick to do. If you look at what this law is saying you will realise that it means that adding the exponents of two exponential numbers (of the same base) is the same as multiplying the two numbers together. This meant that for certain numbers, there was no need to actually multiply the numbers together in order to find out what their multiple was. This saved mathematicians a lot of time, which they could use to do something more productive.



Exercise: Application using Exponential Law 2: $a^m \times a^n = a^{m+n}$

1. $x^2 \cdot x^5 = x^7$
2. $2x^3y \times 5x^2y^7 = 10x^5y^8$
3. $2^3 \cdot 2^4 = 2^7$ [Take note that the base (2) stays the same.]
4. $3 \times 3^{2a} \times 3^2 = 3^{2a+3}$

4.3.3 Exponential Law 3: $a^{-n} = \frac{1}{a^n}, a \neq 0$

Our definition of exponential notation for a negative exponent shows that

$$\begin{aligned}
 a^{-n} &= 1 \div a \div \dots \div a && (n \text{ times}) && (4.11) \\
 &= \frac{1}{1 \times a \times \dots \times a} && (n \text{ times}) \\
 &= \frac{1}{a^n}
 \end{aligned}$$

This means that a minus sign in the exponent is just another way of writing that the whole exponential number is to be divided instead of multiplied.

For example,

$$\begin{aligned}
 2^{-7} &= \frac{1}{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2} \\
 &= \frac{1}{2^7}
 \end{aligned}$$



Exercise: Application using Exponential Law 3: $a^{-n} = \frac{1}{a^n}, a \neq 0$

1. $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
2. $\frac{2^{-2}}{3^2} = \frac{1}{2^2 \cdot 3^2} = \frac{1}{36}$
3. $(\frac{2}{3})^{-3} = (\frac{3}{2})^3 = \frac{27}{8}$

4. $\frac{m}{n^{-4}} = mn^4$
 5. $\frac{a^{-3} \cdot x^4}{a^5 \cdot x^{-2}} = \frac{x^4 \cdot x^2}{a^3 \cdot a^5} = \frac{x^6}{a^8}$
-

4.3.4 Exponential Law 4: $a^m \div a^n = a^{m-n}$

We already realised with law 3 that a minus sign is another way of saying that the exponential number is to be divided instead of multiplied. Law 4 is just a more general way of saying the same thing. We can get this law by just multiplying law 3 by a^m on both sides and using law 2.

$$\begin{aligned} \frac{a^m}{a^n} &= a^m a^{-n} \\ &= a^{m-n} \end{aligned} \quad (4.12)$$

For example,

$$\begin{aligned} 2^7 \div 2^3 &= \frac{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2} \\ &= 2 \times 2 \times 2 \times 2 \\ &= 2^4 \\ &= 2^{7-3} \end{aligned}$$



Exercise: Exponential Law 4: $a^m \div a^n = a^{m-n}$

1. $\frac{a^6}{a^2} = a^{6-2} = a^4$
 2. $\frac{3^2}{3^6} = 3^{2-6} = 3^{-4} = \frac{1}{3^4}$ [Always give final answer with positive index]
 3. $\frac{32a^2}{4a^8} = 8a^{-6} = \frac{8}{a^6}$
 4. $\frac{a^{3x}}{a^4} = a^{3x-4}$
-

4.3.5 Exponential Law 5: $(ab)^n = a^n b^n$

The order in which two real numbers are multiplied together does not matter. Therefore,

$$\begin{aligned} (ab)^n &= a \times b \times a \times b \times \dots \times a \times b \quad (n \text{ times}) \\ &= a \times a \times \dots \times a \quad (n \text{ times}) \\ &\quad \times b \times b \times \dots \times b \quad (n \text{ times}) \\ &= a^n b^n \end{aligned} \quad (4.13)$$

For example,

$$\begin{aligned} (2 \cdot 3)^4 &= (2 \cdot 3) \times (2 \cdot 3) \times (2 \cdot 3) \times (2 \cdot 3) \\ &= (2 \times 2 \times 2 \times 2) \times (3 \times 3 \times 3 \times 3) \\ &= (2^4) \times (3^4) \\ &= 2^4 3^4 \end{aligned}$$



Exercise: Exponential Law 5: $(ab)^n = a^n b^n$

- $(2x^2y)^3 = 2^3 x^{2 \times 3} y^5 = 8x^6 y^5$
- $(\frac{7a}{b^3})^2 = \frac{49a^2}{b^6}$
- $(5a^{n-4})^3 = 125a^{3n-12}$

4.3.6 Exponential Law 6: $(a^m)^n = a^{mn}$

We can find the exponential of an exponential just as well as we can for a number. After all, an exponential number is a real number.

$$\begin{aligned} (a^m)^n &= a^m \times a^m \times \dots \times a^m && (n \text{ times}) \\ &= a \times a \times \dots \times a && (m \times n \text{ times}) \\ &= a^{mn} \end{aligned} \quad (4.14)$$

For example,

$$\begin{aligned} (2^2)^3 &= (2^2) \times (2^2) \times (2^2) \\ &= (2 \times 2) \times (2 \times 2) \times (2 \times 2) \\ &= (2^6) \\ &= 2^{(2 \times 3)} \end{aligned}$$



Exercise: Exponential Law 6: $(a^m)^n = a^{mn}$

- $(x^3)^4 = x^{12}$
- $[(a^4)^3]^2 = a^{24}$
- $(3^{n+3})^2 = 3^{2n+6}$



Worked Example 1: Simplifying indices

Question: Simplify: $\frac{5^{2x-1} \cdot 9^{x-2}}{15^{2x-3}}$

Answer

Step 1 : Factorise all bases into prime factors:

$$\begin{aligned} &= \frac{5^{2x-1} \cdot (3^2)^{x-2}}{(5 \cdot 3)^{2x-3}} \\ &= \frac{5^{2x-1} \cdot 3^{2x-4}}{5^{2x-3} \cdot 3^{2x-3}} \end{aligned}$$

Step 2 : Add and subtract the indices of the same bases as per laws 2 and 4:

$$= 5^{2x-1-2x-3} \cdot 3^{2x-4-2x+3}$$

$$= 5^2 \cdot 3^{-1}$$

Step 3 : Write simplified answer with positive indices:

$$= \frac{25}{3}$$

Activity :: Investigation : Exponential Numbers

Match the answers to the questions, by filling in the correct answer into the **Answer** column. Possible answers are: $\frac{3}{2}$, 1, -1, -3, 8.

Question	Answer
2^3	
7^{3-3}	
$(\frac{2}{3})^{-1}$	
8^{7-6}	
$(-3)^{-1}$	
$(-1)^{23}$	

4.4 End of Chapter Exercises

1. Simplify as far as possible:

(a) 302^0 (b) 1^0 (c) $(xyz)^0$ (d) $[(3x^4y^7z^{12})^5(-5x^9y^3z^4)^2]^0$
 (e) $(2x)^3$ (f) $(-2x)^3$ (g) $(2x)^4$ (h) $(-2x)^4$

2. Simplify without using a calculator. Leave your answers with positive exponents.

(a) $\frac{3x^{-3}}{(3x)^2}$

(b) $5x^0 + 8^{-2} - (\frac{1}{2})^{-2} \cdot 1^x$

(c) $\frac{5^{b-3}}{5^{b+1}}$

3. Simplify, showing all steps:

(a) $\frac{2^{a-2} \cdot 3^{a+3}}{6^a}$

(b) $\frac{a^{2m+n+p}}{a^{m+n+p} \cdot a^m}$

(c) $\frac{3^n \cdot 9^{n-3}}{27^{n-1}}$

(d) $(\frac{2x^{2a}}{y^{-b}})^3$

$$(e) \frac{2^{3x-1} \cdot 8^{x+1}}{4^{2x-2}}$$

$$(f) \frac{6^{2x} \cdot 11^{2x}}{2^{2x-1} \cdot 3^{2x}}$$

4. Simplify, without using a calculator:

$$(a) \frac{(-3)^{-3} \cdot (-3)^2}{(-3)^{-4}}$$

$$(b) (3^{-1} + 2^{-1})^{-1}$$

$$(c) \frac{9^{n-1} \cdot 27^{3-2n}}{81^{2-n}}$$

$$(d) \frac{2^{3n+2} \cdot 8^{n-3}}{4^{3n-2}}$$

Appendix A

GNU Free Documentation License

Version 1.2, November 2002

Copyright © 2000,2001,2002 Free Software Foundation, Inc.

59 Temple Place, Suite 330, Boston, MA 02111-1307 USA

Everyone is permitted to copy and distribute verbatim copies of this license document, but changing it is not allowed.

PREAMBLE

The purpose of this License is to make a manual, textbook, or other functional and useful document “free” in the sense of freedom: to assure everyone the effective freedom to copy and redistribute it, with or without modifying it, either commercially or non-commercially. Secondly, this License preserves for the author and publisher a way to get credit for their work, while not being considered responsible for modifications made by others.

This License is a kind of “copyleft”, which means that derivative works of the document must themselves be free in the same sense. It complements the GNU General Public License, which is a copyleft license designed for free software.

We have designed this License in order to use it for manuals for free software, because free software needs free documentation: a free program should come with manuals providing the same freedoms that the software does. But this License is not limited to software manuals; it can be used for any textual work, regardless of subject matter or whether it is published as a printed book. We recommend this License principally for works whose purpose is instruction or reference.

APPLICABILITY AND DEFINITIONS

This License applies to any manual or other work, in any medium, that contains a notice placed by the copyright holder saying it can be distributed under the terms of this License. Such a notice grants a world-wide, royalty-free license, unlimited in duration, to use that work under the conditions stated herein. The “Document”, below, refers to any such manual or work. Any member of the public is a licensee, and is addressed as “you”. You accept the license if you copy, modify or distribute the work in a way requiring permission under copyright law.

A “Modified Version” of the Document means any work containing the Document or a portion of it, either copied verbatim, or with modifications and/or translated into another language.

A “Secondary Section” is a named appendix or a front-matter section of the Document that deals exclusively with the relationship of the publishers or authors of the Document to the Document’s overall subject (or to related matters) and contains nothing that could fall directly within that overall subject. (Thus, if the Document is in part a textbook of mathematics, a Secondary Section may not explain any mathematics.) The relationship could be a matter of historical connection with the subject or with related matters, or of legal, commercial, philosophical, ethical or political position regarding them.

The “Invariant Sections” are certain Secondary Sections whose titles are designated, as being those of Invariant Sections, in the notice that says that the Document is released under this License. If a section does not fit the above definition of Secondary then it is not allowed to be designated as Invariant. The Document may contain zero Invariant Sections. If the Document does not identify any Invariant Sections then there are none.

The “Cover Texts” are certain short passages of text that are listed, as Front-Cover Texts or Back-Cover Texts, in the notice that says that the Document is released under this License. A Front-Cover Text may be at most 5 words, and a Back-Cover Text may be at most 25 words.

A “Transparent” copy of the Document means a machine-readable copy, represented in a format whose specification is available to the general public, that is suitable for revising the document straightforwardly with generic text editors or (for images composed of pixels) generic paint programs or (for drawings) some widely available drawing editor, and that is suitable for input to text formatters or for automatic translation to a variety of formats suitable for input to text formatters. A copy made in an otherwise Transparent file format whose markup, or absence of markup, has been arranged to thwart or discourage subsequent modification by readers is not Transparent. An image format is not Transparent if used for any substantial amount of text. A copy that is not “Transparent” is called “Opaque”.

Examples of suitable formats for Transparent copies include plain ASCII without markup, Texinfo input format, \LaTeX input format, SGML or XML using a publicly available DTD and standard-conforming simple HTML, PostScript or PDF designed for human modification. Examples of transparent image formats include PNG, XCF and JPG. Opaque formats include proprietary formats that can be read and edited only by proprietary word processors, SGML or XML for which the DTD and/or processing tools are not generally available, and the machine-generated HTML, PostScript or PDF produced by some word processors for output purposes only.

The “Title Page” means, for a printed book, the title page itself, plus such following pages as are needed to hold, legibly, the material this License requires to appear in the title page. For works in formats which do not have any title page as such, “Title Page” means the text near the most prominent appearance of the work’s title, preceding the beginning of the body of the text.

A section “Entitled XYZ” means a named subunit of the Document whose title either is precisely XYZ or contains XYZ in parentheses following text that translates XYZ in another language. (Here XYZ stands for a specific section name mentioned below, such as “Acknowledgements”, “Dedications”, “Endorsements”, or “History”.) To “Preserve the Title” of such a section when you modify the Document means that it remains a section “Entitled XYZ” according to this definition.

The Document may include Warranty Disclaimers next to the notice which states that this License applies to the Document. These Warranty Disclaimers are considered to be included by reference in this License, but only as regards disclaiming warranties: any other implication that these Warranty Disclaimers may have is void and has no effect on the meaning of this License.

VERBATIM COPYING

You may copy and distribute the Document in any medium, either commercially or non-commercially, provided that this License, the copyright notices, and the license notice saying this License applies to the Document are reproduced in all copies, and that you add no other conditions whatsoever to those of this License. You may not use technical measures to obstruct or control the reading or further copying of the copies you make or distribute. However, you may accept compensation in exchange for copies. If you distribute a large enough number of copies you must also follow the conditions in section A.

You may also lend copies, under the same conditions stated above, and you may publicly display copies.

COPYING IN QUANTITY

If you publish printed copies (or copies in media that commonly have printed covers) of the Document, numbering more than 100, and the Document’s license notice requires Cover Texts,

you must enclose the copies in covers that carry, clearly and legibly, all these Cover Texts: Front-Cover Texts on the front cover, and Back-Cover Texts on the back cover. Both covers must also clearly and legibly identify you as the publisher of these copies. The front cover must present the full title with all words of the title equally prominent and visible. You may add other material on the covers in addition. Copying with changes limited to the covers, as long as they preserve the title of the Document and satisfy these conditions, can be treated as verbatim copying in other respects.

If the required texts for either cover are too voluminous to fit legibly, you should put the first ones listed (as many as fit reasonably) on the actual cover, and continue the rest onto adjacent pages.

If you publish or distribute Opaque copies of the Document numbering more than 100, you must either include a machine-readable Transparent copy along with each Opaque copy, or state in or with each Opaque copy a computer-network location from which the general network-using public has access to download using public-standard network protocols a complete Transparent copy of the Document, free of added material. If you use the latter option, you must take reasonably prudent steps, when you begin distribution of Opaque copies in quantity, to ensure that this Transparent copy will remain thus accessible at the stated location until at least one year after the last time you distribute an Opaque copy (directly or through your agents or retailers) of that edition to the public.

It is requested, but not required, that you contact the authors of the Document well before redistributing any large number of copies, to give them a chance to provide you with an updated version of the Document.

MODIFICATIONS

You may copy and distribute a Modified Version of the Document under the conditions of sections A and A above, provided that you release the Modified Version under precisely this License, with the Modified Version filling the role of the Document, thus licensing distribution and modification of the Modified Version to whoever possesses a copy of it. In addition, you must do these things in the Modified Version:

1. Use in the Title Page (and on the covers, if any) a title distinct from that of the Document, and from those of previous versions (which should, if there were any, be listed in the History section of the Document). You may use the same title as a previous version if the original publisher of that version gives permission.
2. List on the Title Page, as authors, one or more persons or entities responsible for authorship of the modifications in the Modified Version, together with at least five of the principal authors of the Document (all of its principal authors, if it has fewer than five), unless they release you from this requirement.
3. State on the Title page the name of the publisher of the Modified Version, as the publisher.
4. Preserve all the copyright notices of the Document.
5. Add an appropriate copyright notice for your modifications adjacent to the other copyright notices.
6. Include, immediately after the copyright notices, a license notice giving the public permission to use the Modified Version under the terms of this License, in the form shown in the Addendum below.
7. Preserve in that license notice the full lists of Invariant Sections and required Cover Texts given in the Document's license notice.
8. Include an unaltered copy of this License.
9. Preserve the section Entitled "History", Preserve its Title, and add to it an item stating at least the title, year, new authors, and publisher of the Modified Version as given on the Title Page. If there is no section Entitled "History" in the Document, create one stating the title, year, authors, and publisher of the Document as given on its Title Page, then add an item describing the Modified Version as stated in the previous sentence.

10. Preserve the network location, if any, given in the Document for public access to a Transparent copy of the Document, and likewise the network locations given in the Document for previous versions it was based on. These may be placed in the “History” section. You may omit a network location for a work that was published at least four years before the Document itself, or if the original publisher of the version it refers to gives permission.
11. For any section Entitled “Acknowledgements” or “Dedications”, Preserve the Title of the section, and preserve in the section all the substance and tone of each of the contributor acknowledgements and/or dedications given therein.
12. Preserve all the Invariant Sections of the Document, unaltered in their text and in their titles. Section numbers or the equivalent are not considered part of the section titles.
13. Delete any section Entitled “Endorsements”. Such a section may not be included in the Modified Version.
14. Do not re-title any existing section to be Entitled “Endorsements” or to conflict in title with any Invariant Section.
15. Preserve any Warranty Disclaimers.

If the Modified Version includes new front-matter sections or appendices that qualify as Secondary Sections and contain no material copied from the Document, you may at your option designate some or all of these sections as invariant. To do this, add their titles to the list of Invariant Sections in the Modified Version’s license notice. These titles must be distinct from any other section titles.

You may add a section Entitled “Endorsements”, provided it contains nothing but endorsements of your Modified Version by various parties—for example, statements of peer review or that the text has been approved by an organisation as the authoritative definition of a standard.

You may add a passage of up to five words as a Front-Cover Text, and a passage of up to 25 words as a Back-Cover Text, to the end of the list of Cover Texts in the Modified Version. Only one passage of Front-Cover Text and one of Back-Cover Text may be added by (or through arrangements made by) any one entity. If the Document already includes a cover text for the same cover, previously added by you or by arrangement made by the same entity you are acting on behalf of, you may not add another; but you may replace the old one, on explicit permission from the previous publisher that added the old one.

The author(s) and publisher(s) of the Document do not by this License give permission to use their names for publicity for or to assert or imply endorsement of any Modified Version.

COMBINING DOCUMENTS

You may combine the Document with other documents released under this License, under the terms defined in section A above for modified versions, provided that you include in the combination all of the Invariant Sections of all of the original documents, unmodified, and list them all as Invariant Sections of your combined work in its license notice, and that you preserve all their Warranty Disclaimers.

The combined work need only contain one copy of this License, and multiple identical Invariant Sections may be replaced with a single copy. If there are multiple Invariant Sections with the same name but different contents, make the title of each such section unique by adding at the end of it, in parentheses, the name of the original author or publisher of that section if known, or else a unique number. Make the same adjustment to the section titles in the list of Invariant Sections in the license notice of the combined work.

In the combination, you must combine any sections Entitled “History” in the various original documents, forming one section Entitled “History”; likewise combine any sections Entitled “Acknowledgements”, and any sections Entitled “Dedications”. You must delete all sections Entitled “Endorsements”.

COLLECTIONS OF DOCUMENTS

You may make a collection consisting of the Document and other documents released under this License, and replace the individual copies of this License in the various documents with a single copy that is included in the collection, provided that you follow the rules of this License for verbatim copying of each of the documents in all other respects.

You may extract a single document from such a collection, and distribute it individually under this License, provided you insert a copy of this License into the extracted document, and follow this License in all other respects regarding verbatim copying of that document.

AGGREGATION WITH INDEPENDENT WORKS

A compilation of the Document or its derivatives with other separate and independent documents or works, in or on a volume of a storage or distribution medium, is called an “aggregate” if the copyright resulting from the compilation is not used to limit the legal rights of the compilation’s users beyond what the individual works permit. When the Document is included in an aggregate, this License does not apply to the other works in the aggregate which are not themselves derivative works of the Document.

If the Cover Text requirement of section A is applicable to these copies of the Document, then if the Document is less than one half of the entire aggregate, the Document’s Cover Texts may be placed on covers that bracket the Document within the aggregate, or the electronic equivalent of covers if the Document is in electronic form. Otherwise they must appear on printed covers that bracket the whole aggregate.

TRANSLATION

Translation is considered a kind of modification, so you may distribute translations of the Document under the terms of section A. Replacing Invariant Sections with translations requires special permission from their copyright holders, but you may include translations of some or all Invariant Sections in addition to the original versions of these Invariant Sections. You may include a translation of this License, and all the license notices in the Document, and any Warranty Disclaimers, provided that you also include the original English version of this License and the original versions of those notices and disclaimers. In case of a disagreement between the translation and the original version of this License or a notice or disclaimer, the original version will prevail.

If a section in the Document is Entitled “Acknowledgements”, “Dedications”, or “History”, the requirement (section A) to Preserve its Title (section A) will typically require changing the actual title.

TERMINATION

You may not copy, modify, sub-license, or distribute the Document except as expressly provided for under this License. Any other attempt to copy, modify, sub-license or distribute the Document is void, and will automatically terminate your rights under this License. However, parties who have received copies, or rights, from you under this License will not have their licenses terminated so long as such parties remain in full compliance.

FUTURE REVISIONS OF THIS LICENSE

The Free Software Foundation may publish new, revised versions of the GNU Free Documentation License from time to time. Such new versions will be similar in spirit to the present version, but may differ in detail to address new problems or concerns. See <http://www.gnu.org/copyleft/>.

Each version of the License is given a distinguishing version number. If the Document specifies that a particular numbered version of this License “or any later version” applies to it, you have the option of following the terms and conditions either of that specified version or of any later version that has been published (not as a draft) by the Free Software Foundation. If the Document does not specify a version number of this License, you may choose any version ever published (not as a draft) by the Free Software Foundation.

ADDENDUM: How to use this License for your documents

To use this License in a document you have written, include a copy of the License in the document and put the following copyright and license notices just after the title page:

Copyright © YEAR YOUR NAME. Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled “GNU Free Documentation License”.

If you have Invariant Sections, Front-Cover Texts and Back-Cover Texts, replace the “with...Texts.” line with this:

with the Invariant Sections being LIST THEIR TITLES, with the Front-Cover Texts being LIST, and with the Back-Cover Texts being LIST.

If you have Invariant Sections without Cover Texts, or some other combination of the three, merge those two alternatives to suit the situation.

If your document contains nontrivial examples of program code, we recommend releasing these examples in parallel under your choice of free software license, such as the GNU General Public License, to permit their use in free software.