FHSST Authors

The Free High School Science Texts: Textbooks for High School Students Studying the Sciences Mathematics Grades 10 - 12

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Contents

Chapter 22

Solving Quadratic Equations - Grade 11

22.1 Introduction

In grade 10, the basics of solving linear equations, quadratic equations, exponential equations and linear inequalities were studied. This chapter extends on that work. We look at different methods of solving quadratic equations.

22.2 Solution by Factorisation

The solving of quadratic equations by factorisation was discussed in Grade 10. Here is an example to remind you of what is involved.

Worked Example 103: Solution of Quadratic Equations

Question: Solve the equation $2x^2 - 5x - 12 = 0$.

Answer

Step 1 : Determine whether the equation has common factors

This equation has no common factors.

Step 2 : Determine if the equation is in the form $ax^2 + bx + c$ with $a > 0$

The equation is in the required form, with $a = 2$, $b = -5$ and $c = -12$.

Step 3 : Factorise the quadratic

$2x^2 - 5x - 12$ has factors of the form:

$$(2x + s)(x + v)$$

with s and v constants to be determined. This multiplies out to

$$2x^2 + (s + 2v)x + sv$$

We see that $sv = -12$ and $s + 2v = -5$. This is a set of simultaneous equations in s and v, but it is easy to solve numerically. All the options for s and v are considered below.

s	v	$s+2v$
2	-6	-10
-2	6	10
3	-4	-5
-3	4	5
4	-3	-2
-4	3	2
6	-2	2
-6	2	-2

We see that the combination $s = 3$ and $v = -4$ gives $s + 2v = -5$.

Step 4 : Write the equation with factors

$$(2x + 3)(x - 4) = 0$$

Step 5 : Solve the equation

If two brackets are multiplied together and give 0, then one of the brackets must be 0, therefore

$$2x + 3 = 0$$

or

$$x - 4 = 0$$

Therefore, $x = -\frac{3}{2}$ or $x = 4$

Step 6 : Write the final answer

The solutions to $2x^2 - 5x - 12 = 0$ are $x = -\frac{3}{2}$ or $x = 4$.

It is important to remember that a quadratic equation has to be in the form $ax^2 + bx + c = 0$ before one can solve it using these methods.

Worked Example 104: Solving quadratic equation by factorisation

Question: Solve for a: $a(a - 3) = 10$

Answer

Step 1 : Rewrite the equation in the form $ax^2 + bx + c = 0$

Remove the brackets and move all terms to one side.

$$a^2 - 3a - 10 = 0$$

Step 2 : Factorise the trinomial

$$(a + 2)(a - 5) = 0$$

Step 3 : Solve the equation

$$a + 2 = 0$$

or

$$a - 5 = 0$$

Solve the two linear equations and check the solutions in the original equation.

Step 4 : Write the final answer

Therefore, $a = -2$ or $a = 5$

Worked Example 105: Solving fractions that lead to a quadratic equation

Question: Solve for b: $\frac{3b}{b+2} + 1 = \frac{4}{b+1}$

Answer

Step 1 : Put both sides over the LCM

$$\frac{3b(b+1)+(b+2)(b+1)}{(b+2)(b+1)} = \frac{4(b+2)}{(b+2)(b+1)}$$

Step 2 : Determine the restrictions

The denominators are the same, therefore the numerators must be the same.
However, $b \neq -2$ and $b \neq -1$

Step 3 : Simplify equation to the standard form

$$\begin{aligned} 3b^2+3b+b^2+3b+2 &= 4b+8 \\ 4b^2+2b-6 &= 0 \\ 2b^2+b-3 &= 0 \end{aligned}$$

Step 4 : Factorise the trinomial and solve the equation

$$\begin{aligned} (2b+3)(b-1) &= 0 \\ 2b+3=0 \quad &or \quad b-1=0 \\ b=\frac{-3}{2} \quad &or \quad b=1 \end{aligned}$$

Step 5 : Check solutions in original equation

Both solutions are valid

Therefore, $b = \frac{-3}{2}$ or $b = 1$

Exercise: Solution by Factorisation

Solve the following quadratic equations by factorisation. Some answers may be left in surd form.

1. $2y^2 - 61 = 101$
2. $2y^2 - 10 = 0$
3. $y^2 - 4 = 10$
4. $2y^2 - 8 = 28$
5. $7y^2 = 28$
6. $y^2 + 28 = 100$
7. $7y^2 + 14y = 0$
8. $12y^2 + 24y + 12 = 0$
9. $16y^2 - 400 = 0$
10. $y^2 - 5y + 6 = 0$
11. $y^2 + 5y - 36 = 0$
12. $y^2 + 2y = 8$
13. $-y^2 - 11y - 24 = 0$
14. $13y - 42 = y^2$

15. $y^2 + 9y + 14 = 0$
16. $y^2 - 5ky + 4k^2 = 0$
17. $y(2y + 1) = 15$
18. $\frac{5y}{y-2} + \frac{3}{y} + 2 = \frac{-6}{y^2-2y}$
19. $\frac{y-2}{y+1} = \frac{2y+1}{y-7}$

22.3 Solution by Completing the Square

We have seen that expressions of the form:

$$a^2x^2 - b^2$$

are known as differences of squares and can be factorised as follows:

$$(ax - b)(ax + b).$$

This simple factorisation leads to another technique to solve quadratic equations known as *completing the square*.

We demonstrate with a simple example, by trying to solve for x in:

$$x^2 - 2x - 1 = 0. \tag{22.1}$$

We cannot easily find factors of this term, but the first two terms look similar to the first two terms of the perfect square:

$$(x - 1)^2 = x^2 - 2x + 1.$$

However, we can cheat and create a perfect square by adding 2 to both sides of the equation in (22.1) as:

$$\begin{aligned} x^2 - 2x - 1 &= 0 \\ x^2 - 2x - 1 + 2 &= 0 + 2 \\ x^2 - 2x + 1 &= 2 \\ (x - 1)^2 &= 2 \\ (x - 1)^2 - 2 &= 0 \end{aligned}$$

Now we know that:

$$2 = (\sqrt{2})^2$$

which means that:

$$(x - 1)^2 - 2$$

is a difference of squares. Therefore we can write:

$$(x - 1)^2 - 2 = [(x - 1) - \sqrt{2}][(x - 1) + \sqrt{2}] = 0.$$

The solution to $x^2 - 2x - 1 = 0$ is then:

$$(x - 1) - \sqrt{2} = 0$$

or

$$(x - 1) + \sqrt{2} = 0.$$

This means $x = 1 + \sqrt{2}$ or $x = 1 - \sqrt{2}$. This example demonstrates the use of *completing the square* to solve a quadratic equation.

Method: Solving Quadratic Equations by Completing the Square

1. Write the equation in the form $ax^2 + bx + c = 0$. e.g. $x^2 + 2x - 3 = 0$
2. Take the constant over to the right hand side of the equation. e.g. $x^2 + 2x = 3$
3. If necessary, make the coefficient of the x^2 term $= 1$, by dividing through by the existing coefficient.
4. Take half the coefficient of the x term, square it and add it to both sides of the equation. e.g. in $x^2 + 2x = 3$, half of the x term is 1. $1^1 = 1$. Therefore we add 1 to both sides to get: $x^2 + 2x + 1 = 3 + 1$.
5. Write the left hand side as a perfect square: $(x + 1)^2 - 4 = 0$
6. You should then be able to factorise the equation in terms of difference of squares and then solve for x: $(x + 1 - 2)(x + 1 + 2) = 0$

Worked Example 106: Solving Quadratic Equations by Completing the Square

Question: Solve:

$$x^2 - 10x - 11 = 0$$

by completing the square

Answer

Step 1 : Write the equation in the form $ax^2 + bx + c = 0$

$$x^2 - 10x - 11 = 0$$

Step 2 : Take the constant over to the right hand side of the equation

$$x^2 - 10x = 11$$

Step 3 : Check that the coefficient of the x^2 term is 1.

The coefficient of the x^2 term is 1.

Step 4 : Take half the coefficient of the x term, square it and add it to both sides

The coefficient of the x term is -10. $\frac{(-10)}{2} = -5$. $(-5)^2 = 25$. Therefore:

$$x^2 - 10x + 25 = 11 + 25$$

Step 5 : Write the left hand side as a perfect square

$$(x - 5)^2 - 36 = 0$$

Step 6 : Factorise equation as difference of squares

$$(x - 5)^2 - 36 = 0$$

$$[(x - 5) + 6][(x - 5) - 6] = 0$$

Step 7 : Solve for the unknown value

$$\begin{aligned} [x + 1][x - 11] &= 0 \\ \therefore x = -1 \quad &\text{or} \quad x = 11 \end{aligned}$$

Worked Example 107: Solving Quadratic Equations by Completing the Square

Question: Solve:

$$2x^2 - 8x - 16 = 0$$

by completing the square

Answer

Step 1 : Write the equation in the form $ax^2 + bx + c = 0$

$$2x^2 - 8x - 16 = 0$$

Step 2 : Take the constant over to the right hand side of the equation

$$2x^2 - 8x = 16$$

Step 3 : Check that the coefficient of the x^2 term is 1.

The coefficient of the x^2 term is 2. Therefore, divide both sides by 2:

$$x^2 - 4x = 8$$

Step 4 : Take half the coefficient of the x term, square it and add it to both sides

The coefficient of the x term is -4. $\frac{(-4)}{2} = -2$. $(-2)^2 = 4$. Therefore:

$$x^2 - 4x + 4 = 8 + 4$$

Step 5 : Write the left hand side as a perfect square

$$(x - 2)^2 - 12 = 0$$

Step 6 : Factorise equation as difference of squares

$$[(x - 2) + \sqrt{12}][(x - 2) - \sqrt{12}] = 0$$

Step 7 : Solve for the unknown value

$$\begin{aligned} [x - 2 + \sqrt{12}][x - 2 - \sqrt{12}] &= 0 \\ \therefore x = 2 - \sqrt{12} \quad &\text{or} \quad x = 2 + \sqrt{12} \end{aligned}$$

Step 8 : The last three steps can also be done in a different the way

Leave left hand side written as a perfect square

$$(x - 2)^2 = 12$$

Step 9 : Take the square root on both sides of the equation

$$x - 2 = \pm\sqrt{12}$$

Step 10 : Solve for x

Therefore $x = 2 - \sqrt{12}$ or $x = 2 + \sqrt{12}$

Compare to answer in step 7.

Exercise: Solution by Completing the Square

Solve the following equations by completing the square:

1. $x^2 + 10x - 2 = 0$
2. $x^2 + 4x + 3 = 0$
3. $x^2 + 8x - 5 = 0$
4. $2x^2 + 12x + 4 = 0$
5. $x^2 + 5x + 9 = 0$
6. $x^2 + 16x + 10 = 0$
7. $3x^2 + 6x - 2 = 0$
8. $z^2 + 8z - 6 = 0$
9. $2z^2 - 11z = 0$
10. $5 + 4z - z^2 = 0$

22.4 Solution by the Quadratic Formula

It is not always possible to solve a quadratic equation by factorising and it is lengthy and tedious to solve a quadratic equations by completing the square. In these situations, you can use the *quadratic formula* that gives the solutions to any quadratic equation.

Consider the general form of the quadratic function:

$$f(x) = ax^2 + bx + c.$$

Factor out the a to get:

$$f(x) = a(x^2 + \frac{b}{a}x + \frac{c}{a}). \tag{22.2}$$

Now we need to do some detective work to figure out how to turn (22.2) into a perfect square plus some extra terms. We know that for a perfect square:

$$(m + n)^2 = m^2 + 2mn + n^2$$

and

$$(m - n)^2 = m^2 - 2mn + n^2$$

The key is the middle term, which is $2\times$ the first term $\times$ the second term. In (22.2), we know that the first term is x so $2\times$ the second term is $\frac{b}{a}$. This means that the second term is $\frac{b}{2a}$. So,

$$(x + \frac{b}{2a})^2 = x^2 + 2\frac{b}{2a}x + (\frac{b}{2a})^2.$$

In general if you add a quantity and subtract the same quantity, nothing has changed. This means if we add and subtract $\left(\frac{b}{2a}\right)^2$ from the right hand side of (22.2) we will get:

$$\begin{aligned}
f(x) &= a(x^2 + \frac{b}{a}x + \frac{c}{a}) && (22.3)\\
&= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) && (22.4)\\
&= a\left(\left[x + \left(\frac{b}{2a}\right)\right]^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) && (22.5)\\
&= a\left(\left[x + \left(\frac{b}{2a}\right)\right]^2\right) + c - \frac{b^2}{4a} && (22.6)
\end{aligned}$$

We set $f(x) = 0$ to find its roots, which yields:

$$a(x + \frac{b}{2a})^2 = \frac{b^2}{4a} - c \tag{22.7}$$

Now dividing by a and taking the square root of both sides gives the expression

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \qquad (22.8)$$

Finally, solving for x implies that

$$\begin{aligned} x &= -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \\ &= -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \end{aligned}$$

which can be further simplified to:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (22.9)$$

These are the solutions to the quadratic equation. Notice that there are two solutions in general, but these may not always exists (depending on the sign of the expression $b^2 - 4ac$ under the square root). These solutions are also called the *roots* of the quadratic equation.

Worked Example 108: Using the quadratic formula

Question: Solve for the roots of the function $f(x) = 2x^2 + 3x - 7$.
Answer
Step 1 : Determine whether the equation can be factorised
The expression cannot be factorised. Therefore, the general quadratic formula must be used.
Step 2 : Identify the coefficients in the equation for use in the formula
From the equation:

$$a = 2$$

$$b = 3$$

$$c = -7$$

Step 3 : Apply the quadratic formula
Always write down the formula first and then substitute the values of a, b and c.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && (22.10) \\ &= \frac{-(3) \pm \sqrt{(3)^2 - 4(2)(-7)}}{2(2)} && (22.11) \\ &= \frac{-3 \pm \sqrt{56}}{4} && (22.12) \\ &= \frac{-3 \pm 2\sqrt{14}}{4} && (22.13) \end{aligned}$$

Step 4 : Write the final answer
The two roots of $f(x) = 2x^2 + 3x - 7$ are $x = \frac{-3+2\sqrt{14}}{4}$ and $\frac{-3-2\sqrt{14}}{4}$.

Worked Example 109: Using the quadratic formula but no solution

Question: Solve for the solutions to the quadratic equation $x^2 - 5x + 8 = 0$.

Answer

Step 1 : Determine whether the equation can be factorised

The expression cannot be factorised. Therefore, the general quadratic formula must be used.

Step 2 : Identify the coefficients in the equation for use in the formula

From the equation:

$$a = 1$$
$$b = -5$$
$$c = 8$$

Step 3 : Apply the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (22.14)$$
$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(8)}}{2(1)} \quad (22.15)$$
$$= \frac{5 \pm \sqrt{-7}}{2} \quad (22.16)$$
(22.17)

Step 4 : Write the final answer

Since the expression under the square root is negative these are not real solutions ($\sqrt{-7}$ is not a real number). Therefore there are no real solutions to the quadratic equation $x^2 - 5x + 8 = 0$. This means that the graph of the quadratic function $f(x) = x^2 - 5x + 8$ has no x-intercepts, but that the entire graph lies above the x-axis.

?

Exercise: Solution by the Quadratic Formula

Solve for t using the quadratic formula.

1. $3t^2 + t - 4 = 0$
2. $t^2 - 5t + 9 = 0$
3. $2t^2 + 6t + 5 = 0$
4. $4t^2 + 2t + 2 = 0$
5. $-3t^2 + 5t - 8 = 0$
6. $-5t^2 + 3t - 3 = 0$
7. $t^2 - 4t + 2 = 0$
8. $9t^2 - 7t - 9 = 0$
9. $2t^2 + 3t + 2 = 0$
10. $t^2 + t + 1 = 0$

!

Important:

- In all the examples done so far, the solutions were left in surd form. Answers can also be given in decimal form, using the calculator. Read the instructions when answering questions in a test or exam whether to leave answers in surd form, or in decimal form to an appropriate number of decimal places.
- Completing the square as a method to solve a quadratic equation is only done when specifically asked.

Exercise: Mixed Exercises

Solve the quadratic equations by either factorisation, completing the square or by using the quadratic formula:

- Always try to factorise first, then use the formula if the trinomial cannot be factorised.
- Do some of them by completing the square and then compare answers to those done using the other methods.

1. $24y^2 + 61y - 8 = 0$
2. $-8y^2 - 16y + 42 = 0$
3. $-9y^2 + 24y - 12 = 0$
4. $-5y^2 + 0y + 5 = 0$
5. $-3y^2 + 15y - 12 = 0$
6. $49y^2 + 0y - 25 = 0$
7. $-12y^2 + 66y - 72 = 0$
8. $-40y^2 + 58y - 12 = 0$
9. $-24y^2 + 37y + 72 = 0$
10. $6y^2 + 7y - 24 = 0$
11. $2y^2 - 5y - 3 = 0$
12. $-18y^2 - 55y - 25 = 0$
13. $-25y^2 + 25y - 4 = 0$
14. $-32y^2 + 24y + 8 = 0$
15. $9y^2 - 13y - 10 = 0$
16. $35y^2 - 8y - 3 = 0$
17. $-81y^2 - 99y - 18 = 0$
18. $14y^2 - 81y + 81 = 0$
19. $-4y^2 - 41y - 45 = 0$
20. $16y^2 + 20y - 36 = 0$
21. $42y^2 + 104y + 64 = 0$
22. $9y^2 - 76y + 32 = 0$
23. $-54y^2 + 21y + 3 = 0$
24. $36y^2 + 44y + 8 = 0$
25. $64y^2 + 96y + 36 = 0$
26. $12y^2 - 22y - 14 = 0$
27. $16y^2 + 0y - 81 = 0$
28. $3y^2 + 10y - 48 = 0$
29. $-4y^2 + 8y - 3 = 0$
30. $-5y^2 - 26y + 63 = 0$
31. $x^2 - 70 = 11$
32. $2x^2 - 30 = 2$
33. $x^2 - 16 = 2 - x^2$
34. $2y^2 - 98 = 0$
35. $5y^2 - 10 = 115$
36. $5y^2 - 5 = 19 - y^2$

22.5 Finding an equation when you know its roots

We have mentioned before that the *roots* of a quadratic equation are the solutions or answers you get from solving the quadatic equation. Working back from the answers, will take you to an equation.

Worked Example 110: Find an equation when roots are given

Question: Find an equation with roots 13 and -5

Answer

Step 1 : Write down as the product of two brackets

The step before giving the solutions would be:

$$(x - 13)(x + 5) = 0$$

Notice that the signs in the brackets are opposite of the given roots.

Step 2 : Remove brackets

$$x^2 - 8x - 65 = 0$$

Of course, there would be other possibilities as well when each term on each side of the *equal to sign* is multiplied by a constant.

Worked Example 111: Fraction roots

Question: Find an equation with roots $-\frac{3}{2}$ and 4

Answer

Step 1 : Product of two brackets

Notice that if $x = -\frac{3}{2}$ then $2x + 3 = 0$

Therefore the two brackets will be:

$$(2x + 3)(x - 4) = 0$$

Step 2 : Remove brackets

The equation is:

$$2x^2 - 5x - 12 = 0$$

Extension: Theory of Quadratic Equations - Advanced

This section is not in the syllabus, but it gives one a good understanding about some of the solutions of the quadratic equations.

What is the Discriminant of a Quadratic Equation?

Consider a general quadratic function of the form $f(x) = ax^2 + bx + c$. The *discriminant* is defined as:

$$\Delta = b^2 - 4ac. \tag{22.18}$$

This is the expression under the square root in the formula for the roots of this function. We have already seen that whether the roots exist or not depends on whether this factor Δ is negative or positive.

The Nature of the Roots

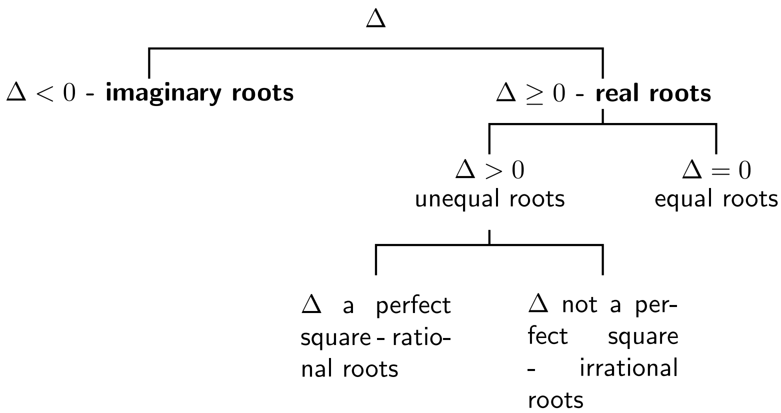
Real Roots ($\Delta \geq 0$)

Consider $\Delta \geq 0$ for some quadratic function $f(x) = ax^2 + bx + c$. In this case there are solutions to the equation $f(x) = 0$ given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a} \tag{22.19}$$

Since the square roots exists (the expression under the square root is non-negative.) These are the roots of the function $f(x)$.

There various possibilities are summarised in the figure below.



Equal Roots ($\Delta = 0$)

If $\Delta = 0$, then the roots are equal and, from the formula, these are given by

$$x = -\frac{b}{2a} \tag{22.20}$$

Unequal Roots ($\Delta > 0$)

There will be 2 unequal roots if $\Delta > 0$. The roots of $f(x)$ are **rational** if Δ is a perfect square (a number which is the square of a rational number), since, in this case, $\sqrt{\Delta}$ is rational. Otherwise, if Δ is not a perfect square, then the roots are **irrational**.

Imaginary Roots ($\Delta < 0$)

If $\Delta < 0$, then the solution to $f(x) = ax^2 + bx + c = 0$ contains the square root of a negative number and therefore there are no real solutions. We therefore say that the roots of $f(x)$ are *imaginary* (the graph of the function $f(x)$ does not intersect the x-axis).

Extension: Theory of Quadratics - advanced exercises

Exercise: From past papers

1. [IEB, Nov. 2001, HG] Given: $x^2 + bx - 2 + k(x^2 + 3x + 2) = 0 \quad (k \neq -1)$
   A Show that the discriminant is given by:
   $$\Delta = k^2 + 6bk + b^2 + 8$$
   B If $b = 0$, discuss the nature of the roots of the equation.
   C If $b = 2$, find the value(s) of k for which the roots are equal.
2. [IEB, Nov. 2002, HG] Show that $k^2x^2 + 2 = kx - x^2$ has non-real roots for all real values for k.
3. [IEB, Nov. 2003, HG] The equation $x^2 + 12x = 3kx^2 + 2$ has real roots.
   A Find the largest integral value of k.
   B Find one rational value of k, for which the above equation has rational roots.
4. [IEB, Nov. 2003, HG] In the quadratic equation $px^2 + qx + r = 0$, p, q and r are positive real numbers and form a geometric sequence. Discuss the nature of the roots.
5. [IEB, Nov. 2004, HG] Consider the equation:
   $$k = \frac{x^2 - 4}{2x - 5} \qquad \text{where } x \neq \tfrac{5}{2}$$
   A Find a value of k for which the roots are equal.
   B Find an integer k for which the roots of the equation will be rational and unequal.
6. [IEB, Nov. 2005, HG]
   A Prove that the roots of the equation $x^2 - (a + b)x + ab - p^2 = 0$ are real for all real values of a, b and p.
   B When will the roots of the equation be equal?
7. [IEB, Nov. 2005, HG] If b and c can take on only the values 1, 2 or 3, determine all pairs $(b; c)$ such that $x^2 + bx + c = 0$ has real roots.

22.6 End of Chapter Exercises

1. Solve: $x^2 - x - 1 = 0$ (Give your answer correct to two decimal places.)
2. Solve: $16(x+1) = x^2(x+1)$
3. Solve: $y^2 + 3 + \frac{12}{y^2+3} = 7$ (Hint: Let $y^2 + 3 = k$ and solve for k first and use the answer to solve y.)
4. Solve for x: $2x^4 - 5x^2 - 12 = 0$
5. Solve for x:
   - A $x(x-9) + 14 = 0$
   - B $x^2 - x = 3$ (Show your answer correct to ONE decimal place.)
   - C $x + 2 = \frac{6}{x}$ (correct to 2 decimal places)
   - D $\frac{1}{x+1} + \frac{2x}{x-1} = 1$
6. Solve for x by completing the square: $x^2 - px - 4 = 0$
7. The equation $ax^2 + bx + c = 0$ has roots $x = \frac{2}{3}$ and $x = -4$. Find one set of possible values for a, b and c.
8. The two roots of the equation $4x^2 + px - 9 = 0$ differ by 5. Calculate the value of p.
9. An equation of the form $x^2 + bx + c = 0$ is written on the board. Saskia and Sven copy it down incorrectly. Saskia has a mistake in the constant term and obtains the solutions -4 and 2. Sven has a mistake in the coefficient of x and obtains the solutions 1 and -15. Determine the correct equation that was on the board.
10. Bjorn stumbled across the following formula to solve the quadratic equation $ax^2+bx+c = 0$ in a foreign textbook.

$$x = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}$$

   - A Use this formula to solve the equation:

$$2x^2 + x - 3 = 0$$

   - B Solve the equation again, using factorisation, to see if the formula works for this equation.
   - C Trying to derive this formula to prove that it always works, Bjorn got stuck along the way. His attempt his shown below:

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
a + \frac{b}{x} + \frac{c}{x^2} &= 0 && \text{Divided by } x^2 \text{ where } x \neq 0 \\
\frac{c}{x^2} + \frac{b}{x} + a &= 0 && \text{Rearranged} \\
\frac{1}{x^2} + \frac{b}{cx} + \frac{a}{c} &= 0 && \text{Divided by } c \text{ where } c \neq 0 \\
\frac{1}{x^2} + \frac{b}{cx} &= -\frac{a}{c} && \text{Subtracted } \frac{a}{c} \text{ from both sides} \\
\therefore \frac{1}{x^2} + \frac{b}{cx} &+ \ldots && \text{Got stuck}
\end{aligned}$$

     Complete his derivation.

Appendix A

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