



FHSST Authors

**The Free High School Science Texts:
Textbooks for High School Students
Studying the Sciences
Mathematics
Grades 10 - 12**

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this a continuously evolving resource!

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Chapter 42

Geometry - Grade 12

42.1 Introduction

Activity :: Discussion : Discuss these Research Topics

Research one of the following geometrical ideas and describe it to your group:

1. taxicab geometry,
 2. spherical geometry,
 3. fractals,
 4. the Koch snowflake.
-

42.2 Circle Geometry

42.2.1 Terminology

The following is a recap of terms that are regularly used when referring to circles.

arc An arc is a part of the circumference of a circle.

chord A chord is defined as a straight line joining the ends of an arc.

radius The radius, r , is the distance from the centre of the circle to any point on the circumference.

diameter The diameter, \varnothing , is a special chord that passes through the centre of the circle. The diameter is the straight line from a point on the circumference to another point on the circumference, that passes through the centre of the circle.

segment A segment is the part of the circle that is cut off by a chord. A chord divides a circle into two segments.

tangent A tangent is a line that makes contact with a circle at one point on the circumference. (AB is a tangent to the circle at point P .)

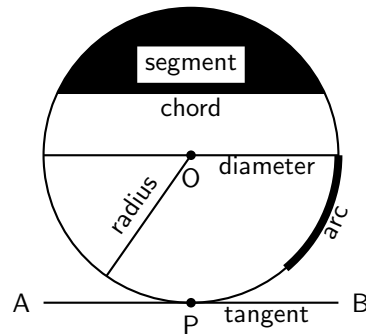


Figure 42.1: Parts of a Circle

42.2.2 Axioms

An axiom is an established or accepted principle. For this section, the following are accepted as axioms.

1. The Theorem of Pythagoras, which states that the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides. In $\triangle ABC$, this means that $AB^2 + BC^2 = AC^2$

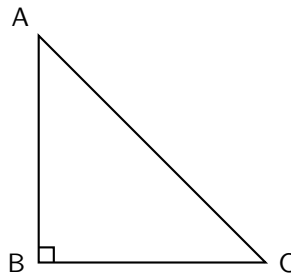


Figure 42.2: A right-angled triangle

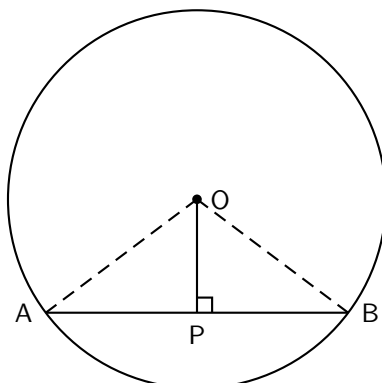
2. A tangent is perpendicular to the radius, drawn at the point of contact with the circle.

42.2.3 Theorems of the Geometry of Circles

A theorem is a general proposition that is not self-evident but is proved by reasoning (these proofs need not be learned for examination purposes).

Theorem 6. *The line drawn from the centre of a circle, perpendicular to a chord, bisects the chord.*

Proof:



Consider a circle, with centre O . Draw a chord AB and draw a perpendicular line from the centre of the circle to intersect the chord at point P .

The aim is to prove that $AP = BP$

1. $\triangle OAP$ and $\triangle OBP$ are right-angled triangles.
2. $OA = OB$ as both of these are radii and OP is common to both triangles.

Apply the Theorem of Pythagoras to each triangle, to get:

$$\begin{aligned} OA^2 &= OP^2 + AP^2 \\ OB^2 &= OP^2 + BP^2 \end{aligned}$$

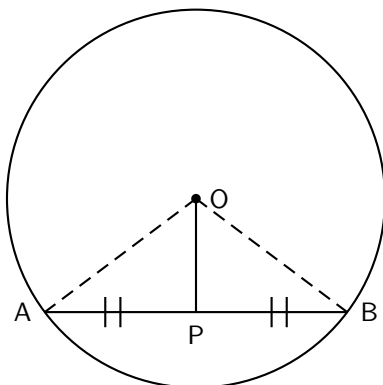
However, $OA = OB$. So,

$$\begin{aligned} OP^2 + AP^2 &= OP^2 + BP^2 \\ \therefore AP^2 &= BP^2 \\ \text{and } AP &= BP \end{aligned}$$

This means that OP bisects AB .

Theorem 7. *The line drawn from the centre of a circle, that bisects a chord, is perpendicular to the chord.*

Proof:



Consider a circle, with centre O . Draw a chord AB and draw a line from the centre of the circle to bisect the chord at point P .

The aim is to prove that $OP \perp AB$

In $\triangle OAP$ and $\triangle OBP$,

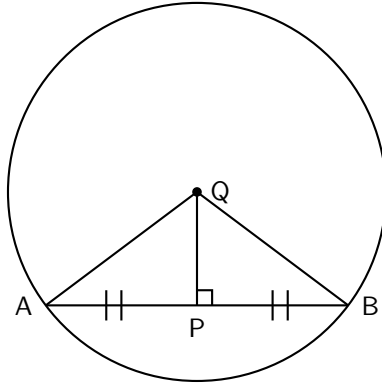
1. $AP = PB$ (given)
2. $OA = OB$ (radii)
3. OP is common to both triangles.

$\therefore \triangle OAP \cong \triangle OBP$ (SSS).

$$\begin{aligned} \hat{O}AP &= \hat{O}BP \\ \hat{O}AP + \hat{O}BP &= 180^\circ \quad (APB \text{ is a str. line}) \\ \therefore \hat{O}AP &= \hat{O}BP = 90^\circ \\ \therefore OP &\perp AB \end{aligned}$$

Theorem 8. *The perpendicular bisector of a chord passes through the centre of the circle.*

Proof:



Consider a circle. Draw a chord AB . Draw a line PQ perpendicular to AB such that PQ bisects AB at point P . Draw lines AQ and BQ .

The aim is to prove that Q is the centre of the circle, by showing that $AQ = BQ$.

In $\triangle OAP$ and $\triangle OBP$,

1. $AP = PB$ (given)
2. $\angle QPA = \angle QPB$ ($QP \perp AB$)
3. QP is common to both triangles.

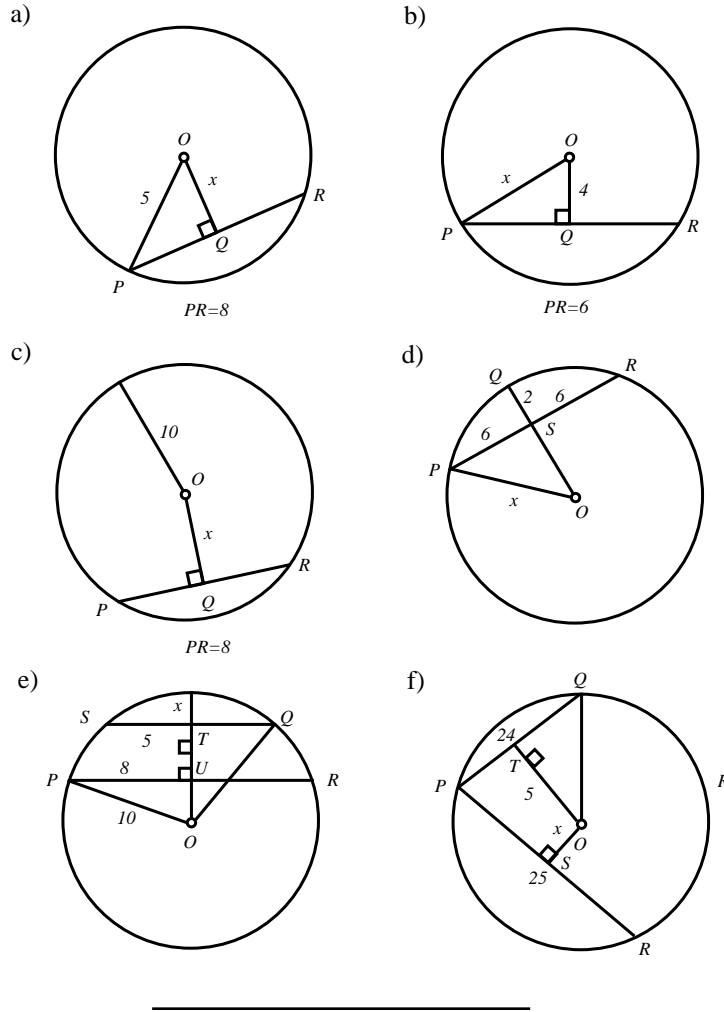
$\therefore \triangle QAP \equiv \triangle QBP$ (SAS).

From this, $QA = QB$. Since the centre of a circle is the only point inside a circle that has points on the circumference at an equal distance from it, Q must be the centre of the circle.



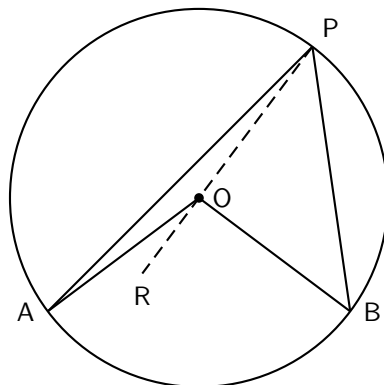
Exercise: Circles I

1. Find the value of x :



Theorem 9. *The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circumference of the circle.*

Proof:



Consider a circle, with centre O and with A and B on the circumference. Draw a chord AB . Draw radii OA and OB . Select any point P on the circumference of the circle. Draw lines PA and PB . Draw PO and extend to R .

The aim is to prove that $\hat{A}OB = 2 \cdot \hat{A}PB$.

$\hat{A}OR = \hat{P}AO + \hat{A}PO$ (exterior angle = sum of interior opp. angles)

But, $\hat{P}AO = \hat{A}PO$ ($\triangle AOP$ is an isosceles \triangle)

$$\therefore \hat{AOR} = 2\hat{APO}$$

Similarly, $\hat{BOR} = 2\hat{BPO}$.

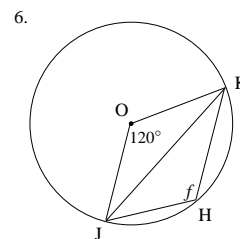
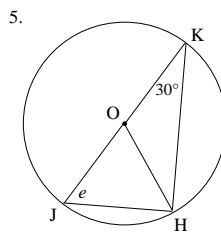
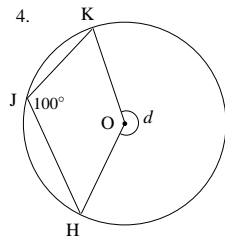
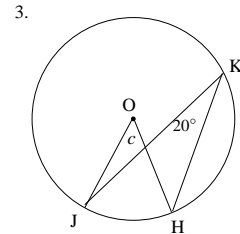
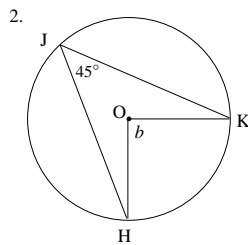
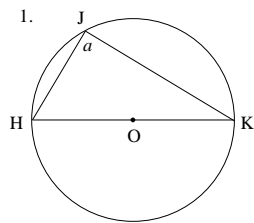
So,

$$\begin{aligned} \hat{AOB} &= \hat{AOR} + \hat{BOR} \\ &= 2\hat{APO} + 2\hat{BPO} \\ &= 2(\hat{APO} + \hat{BPO}) \\ &= 2(\hat{APB}) \end{aligned}$$



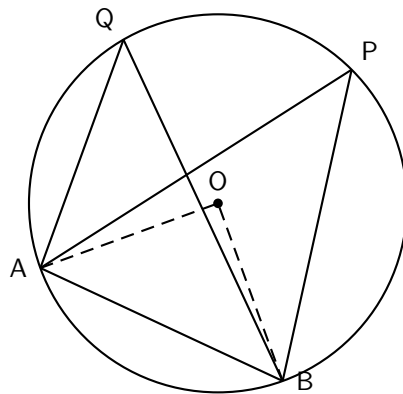
Exercise: Circles II

1. Find the angles (*a* to *f*) indicated in each diagram:



Theorem 10. *The angles subtended by a chord at the circumference of a circle on the same side of the chord are equal.*

Proof:



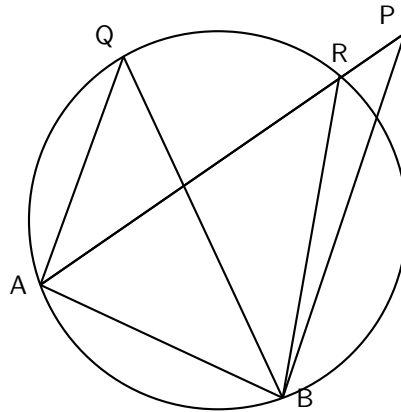
Consider a circle, with centre *O*. Draw a chord *AB*. Select any points *P* and *Q* on the circumference of the circle, such that both *P* and *Q* are on the same side of the chord. Draw lines *PA*, *PB*, *QA* and *QB*.

The aim is to prove that $\hat{AQB} = \hat{APB}$.

$$\begin{aligned} \hat{AOB} &= 2\hat{AQB} \text{ } \angle \text{ at centre} = \text{twice } \angle \text{ at circumference} \\ \text{and } \hat{AOB} &= 2\hat{APB} \text{ } \angle \text{ at centre} = \text{twice } \angle \text{ at circumference} \\ \therefore 2\hat{AQB} &= 2\hat{APB} \\ \therefore \hat{AQB} &= \hat{APB} \end{aligned}$$

Theorem 11. (Converse of Theorem 10) *If a line segment subtends equal angles at two other points on the same side of the line, then these four points lie on a circle.*

Proof:



Consider a line segment AB , that subtends equal angles at points P and Q on the same side of AB .

The aim is to prove that points A , B , P and Q lie on the circumference of a circle.

By contradiction. Assume that point P does not lie on a circle drawn through points A , B and Q . Let the circle cut AP (or AP extended) at point R .

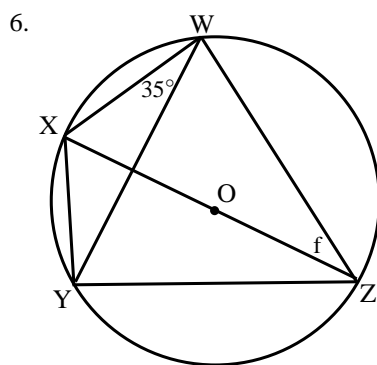
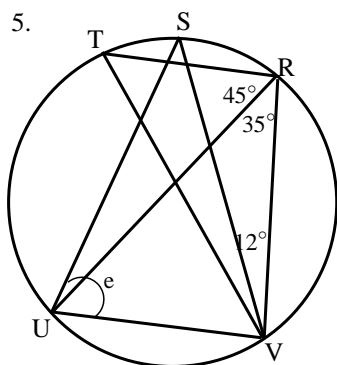
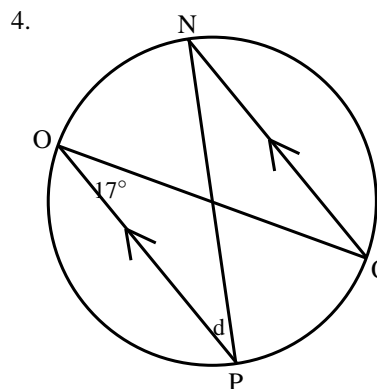
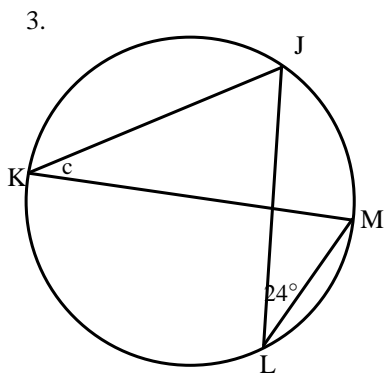
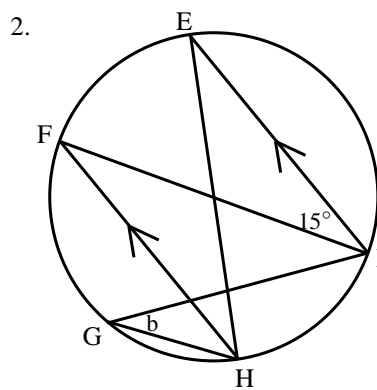
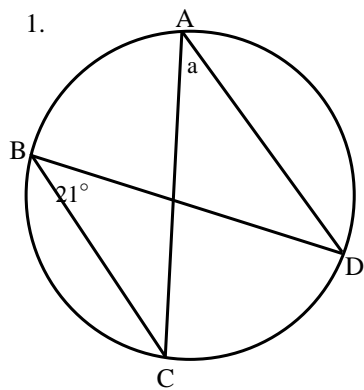
$$\begin{aligned} \hat{AQB} &= \hat{ARB} \text{ } \angle \text{ s on same side of chord} \\ \text{but } \hat{AQB} &= \hat{APB} \text{ (given)} \\ \therefore \hat{ARB} &= \hat{APB} \\ \text{but this cannot be true since } \hat{ARB} &= \hat{APB} + \hat{RBP} \text{ (ext. } \angle \text{ of } \triangle) \end{aligned}$$

\therefore the assumption that the circle does not pass through P , must be false, and A , B , P and Q lie on the circumference of a circle.



Exercise: Circles III

1. Find the values of the unknown letters.

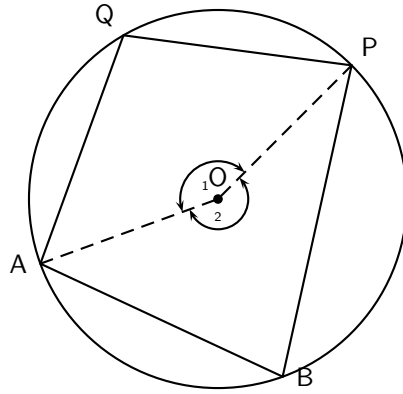


Cyclic Quadrilaterals

Cyclic quadrilaterals are quadrilaterals with all four vertices lying on the circumference of a circle. The vertices of a cyclic quadrilateral are said to be *conyclic*.

Theorem 12. *The opposite angles of a cyclic quadrilateral are supplementary.*

Proof:



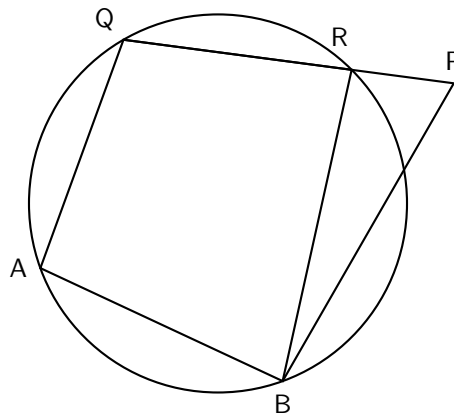
Consider a circle, with centre O . Draw a cyclic quadrilateral $ABPQ$. Draw AO and PO .

The aim is to prove that $\hat{A}BP + \hat{A}QP = 180^\circ$ and $\hat{Q}AB + \hat{Q}PB = 180^\circ$.

$$\begin{aligned} \hat{O}_1 &= 2\hat{A}BP \text{ } \angle\text{s at centre} \\ \hat{O}_2 &= 2\hat{A}QP \text{ } \angle\text{s at centre} \\ \text{But, } \hat{O}_1 + \hat{O}_2 &= 360^\circ \\ \therefore 2\hat{A}BP + 2\hat{A}QP &= 360^\circ \\ \therefore \hat{A}BP + \hat{A}QP &= 180^\circ \\ \text{Similarly, } \hat{Q}AB + \hat{Q}PB &= 180^\circ \end{aligned}$$

Theorem 13. (Converse of Theorem 12) *If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.*

Proof:



Consider a quadrilateral $ABPQ$, such that $\hat{A}BP + \hat{A}QP = 180^\circ$ and $\hat{Q}AB + \hat{Q}PB = 180^\circ$.

The aim is to prove that points A, B, P and Q lie on the circumference of a circle.

By contradiction. Assume that point P does not lie on a circle drawn through points A, B and Q . Let the circle cut AP (or AP extended) at point R . Draw BR .

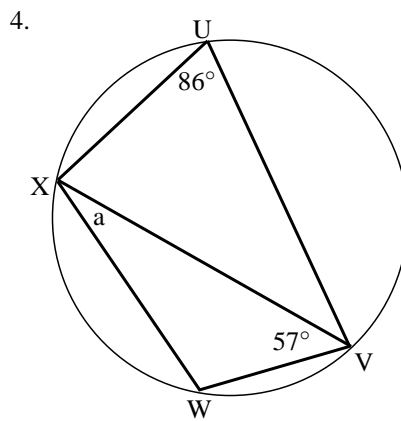
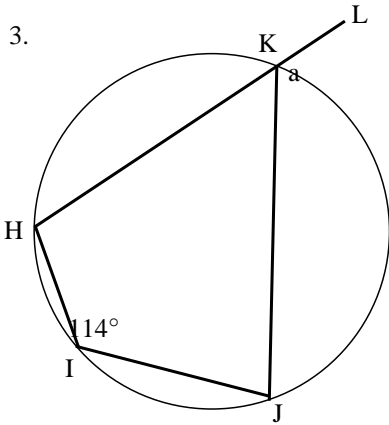
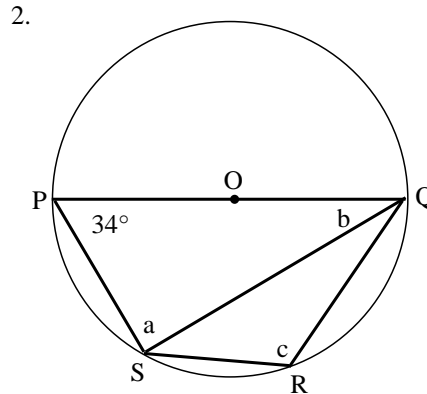
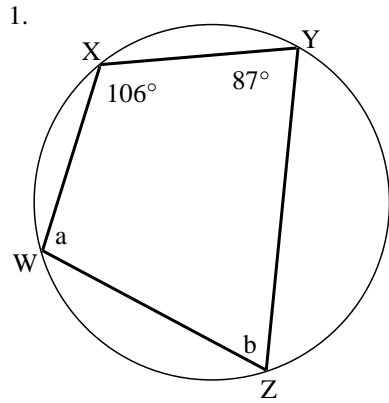
$$\begin{aligned} \hat{Q}AB + \hat{Q}RB &= 180^\circ \text{ opp. } \angle\text{s of cyclic quad.} \\ \text{but } \hat{Q}AB + \hat{Q}PB &= 180^\circ \text{ (given)} \\ \therefore \hat{Q}RB &= \hat{Q}PB \\ \text{but this cannot be true since } \hat{Q}RB &= \hat{Q}PB + \hat{R}BP \text{ (ext. } \angle \text{ of } \triangle) \end{aligned}$$

\therefore the assumption that the circle does not pass through P , must be false, and A, B, P and Q lie on the circumference of a circle and $ABPQ$ is a cyclic quadrilateral.



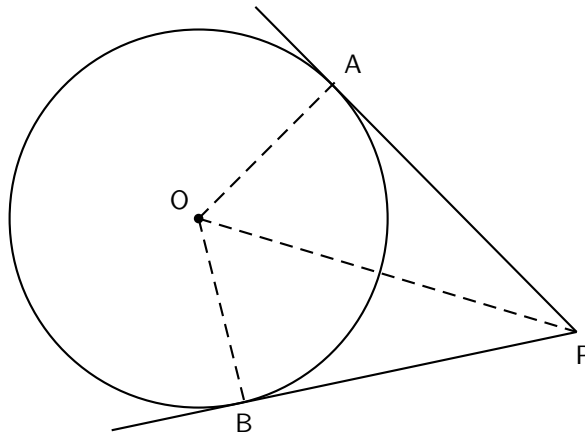
Exercise: Circles IV

1. Find the values of the unknown letters.



Theorem 14. Two tangents drawn to a circle from the same point outside the circle are equal in length.

Proof:



Consider a circle, with centre O . Choose a point P outside the circle. Draw two tangents to the circle from point P , that meet the circle at A and B . Draw lines OA , OB and OP .

The aim is to prove that $AP = BP$.

In $\triangle OAP$ and $\triangle OBP$,

1. $OA = OB$ (radii)
2. $\angle OAP = \angle OPB = 90^\circ$ ($OA \perp AP$ and $OB \perp BP$)
3. OP is common to both triangles.

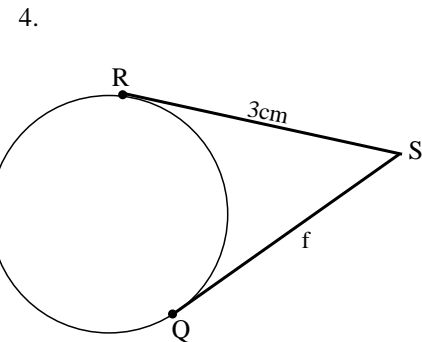
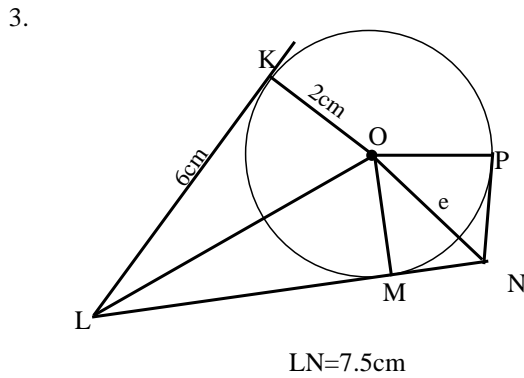
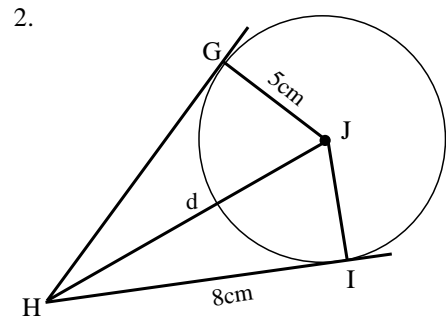
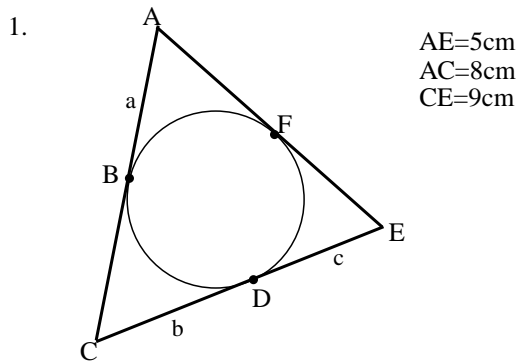
$\triangle OAP \cong \triangle OBP$ (right angle, hypotenuse, side)

$\therefore AP = BP$



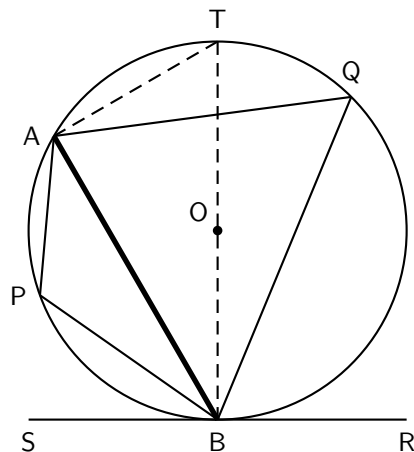
Exercise: Circles V

1. Find the value of the unknown lengths.



Theorem 15. *The angle between a tangent and a chord, drawn at the point of contact of the chord, is equal to the angle which the chord subtends in the alternate segment.*

Proof:



Consider a circle, with centre O . Draw a chord AB and a tangent SR to the circle at point B . Chord AB subtends angles at points P and Q on the minor and major arcs, respectively.

Draw a diameter BT and join A to T .

The aim is to prove that $\hat{APB} = \hat{ABR}$ and $\hat{AQB} = \hat{ABS}$.

First prove that $\hat{AQB} = \hat{ABS}$ as this result is needed to prove that $\hat{APB} = \hat{ABR}$.

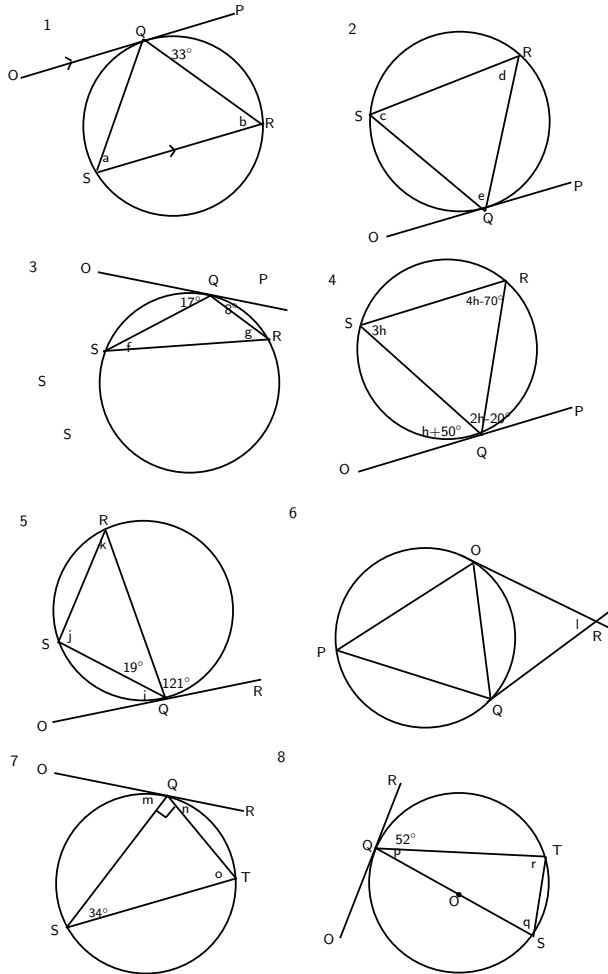
$$\begin{aligned}
 \hat{ABS} + \hat{ABT} &= 90^\circ \quad (TB \perp SR) \\
 \hat{BAT} &= 90^\circ \quad (\angle\text{s at centre}) \\
 \therefore \hat{ABT} + \hat{ATB} &= 90^\circ \quad (\text{sum of angles in } \triangle BAT) \\
 \therefore \hat{ABS} &= \hat{ABT} \\
 \text{However, } \hat{AQB} &= \hat{ATB} \quad (\text{angles subtended by same chord } AB) \\
 \therefore \hat{AQB} &= \hat{ABS} \tag{42.1}
 \end{aligned}$$

$$\begin{aligned}
 \hat{SBQ} + \hat{QBR} &= 180^\circ \quad (SBT \text{ is a str. line}) \\
 \hat{APB} + \hat{AQB} &= 180^\circ \quad (ABPQ \text{ is a cyclic quad.}) \\
 \therefore \hat{SBQ} + \hat{QBR} &= \hat{APB} + \hat{AQB} \\
 \text{From (42.1), } \hat{AQB} &= \hat{ABS} \\
 \therefore \hat{APB} &= \hat{ABR}
 \end{aligned}$$



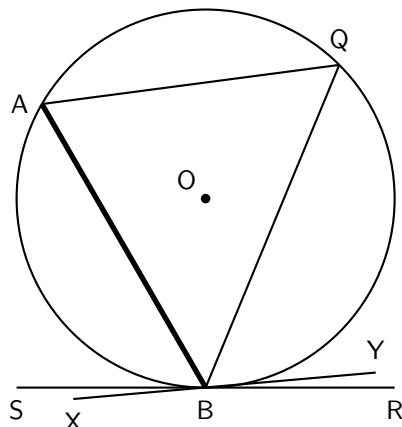
Exercise: Circles VI

1. Find the values of the unknown letters.



Theorem 16. (Converse of 15) If the angle formed between a line, that is drawn through the end point of a chord, and the chord, is equal to the angle subtended by the chord in the alternate segment, then the line is a tangent to the circle.

Proof:



Consider a circle, with centre O and chord AB . Let line SR pass through point B . Chord AB subtends an angle at point Q such that $\hat{A}BS = \hat{A}QB$.

The aim is to prove that SBR is a tangent to the circle.

By contradiction. Assume that SBR is not a tangent to the circle and draw XY such that XY is a tangent to the circle.

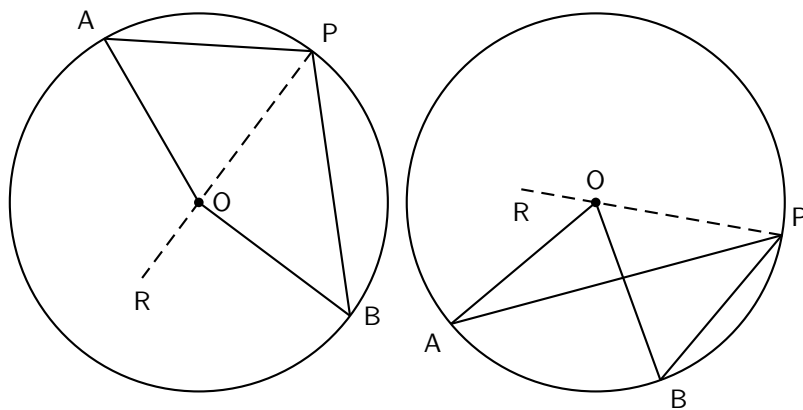
$$\begin{aligned}
 \hat{A}BX &= \hat{A}QB \quad (\text{tan-chord theorem}) \\
 \text{However, } \hat{A}BS &= \hat{A}QB \quad (\text{given}) \\
 \therefore \hat{A}BX &= \hat{A}BS \\
 \text{But since, } \hat{A}BX &= \hat{A}BS + \hat{X}BS \\
 (42.2) \text{ can only be true if, } \hat{X}BS &= 0
 \end{aligned}
 \tag{42.2}$$

If $\hat{X}BS$ is zero, then both XY and SBR coincide and SBR is a tangent to the circle.



Exercise: Applying Theorem 9

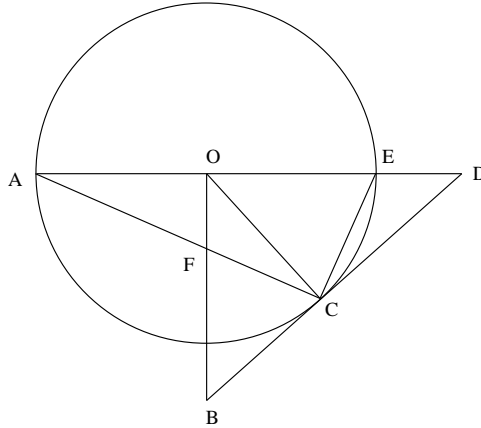
1. Show that Theorem 9 also applies to the following two cases:





Worked Example 190: Circle Geometry I

Question:



BD is a tangent to the circle with centre O .
 $BO \perp AD$.

Prove that:

1. $CFOE$ is a cyclic quadrilateral
2. $FB = BC$
3. $\triangle COE \sim \triangle CBF$
4. $CD^2 = ED \cdot AD$
5. $\frac{OE}{BC} = \frac{CD}{CO}$

Answer

1. **Step 1 :** To show a quadrilateral is cyclic, we need a pair of opposite angles to be supplementary, so let's look for that.

$$\hat{F}OE = 90^\circ \text{ (} BO \perp OD \text{)}$$

$$\hat{F}CE = 90^\circ \text{ (} \angle \text{ subtended by diameter } AE \text{)}$$

$\therefore CFOE$ is a cyclic quadrilateral (opposite \angle 's supplementary)

2. **Step 1 :** Since these two sides are part of a triangle, we are proving that triangle to be isosceles. The easiest way is to show the angles opposite to those sides to be equal.

Let $\hat{O}EC = x$.

$$\therefore \hat{F}CB = x \text{ (} \angle \text{ between tangent } BD \text{ and chord } CE \text{)}$$

$$\therefore \hat{B}FC = x \text{ (exterior } \angle \text{ to cyclic quadrilateral } CFOE \text{)}$$

$$\therefore BF = BC \text{ (sides opposite equal } \angle \text{'s in isosceles } \triangle BFC \text{)}$$

3. **Step 1 :** To show these two triangles similar, we will need 3 equal angles. We already have 3 of the 6 needed angles from the previous question. We need only find the missing 3 angles.

$$\hat{C}BF = 180^\circ - 2x \text{ (sum of } \angle \text{'s in } \triangle BFC \text{)}$$

$$OC = OE \text{ (radii of circle } O \text{)}$$

$$\therefore \hat{E}CO = x \text{ (isosceles } \triangle COE \text{)}$$

$$\therefore \hat{C}OE = 180^\circ - 2x \text{ (sum of } \angle \text{'s in } \triangle COE \text{)}$$

$$\bullet \hat{C}OE = \hat{C}BF$$

$$\bullet \hat{E}CO = \hat{F}CB$$

$$\bullet \hat{O}EC = \hat{F}BC$$

$$\therefore \triangle COE \sim \triangle CBF \text{ (3 } \angle \text{'s equal)}$$

4. **Step 1 :** This relation reminds us of a proportionality relation between similar triangles. So investigate which triangles contain these sides and prove them similar. In this case 3 equal angles works well. Start with one triangle.

In $\triangle EDC$

$$\hat{CED} = 180^\circ - x \text{ (}\angle\text{'s on a straight line } AD\text{)}$$

$$\hat{ECD} = 90^\circ - x \text{ (complementary } \angle\text{'s)}$$

Step 2 : Now look at the angles in the other triangle.

In $\triangle ADC$

$$\hat{ACE} = 180^\circ - x \text{ (sum of } \angle\text{'s } \hat{ACE} \text{ and } \hat{ECO}\text{)}$$

$$\hat{CAD} = 90^\circ - x \text{ (sum of } \angle\text{'s in } \triangle CAE\text{)}$$

Step 3 : The third equal angle is an angle both triangles have in common.

Lastly, $\hat{ADC} = \hat{EDC}$ since they are the same \angle .

Step 4 : Now we know that the triangles are similar and can use the proportionality relation accordingly.

$$\therefore \triangle ADC \sim \triangle CDE \text{ (3 } \angle\text{'s equal)}$$

$$\therefore \frac{ED}{CD} = \frac{CD}{AD}$$

$$\therefore CD^2 = ED \cdot AD$$

5. **Step 1 :** This looks like another proportionality relation with a little twist, since not all sides are contained in 2 triangles. There is a quick observation we can make about the odd side out, OE .

$$OE = CD \text{ (}\triangle OEC \text{ is isosceles)}$$

Step 2 : With this observation we can limit ourselves to proving triangles BOC and ODC similar. Start in one of the triangles.

In $\triangle BCO$

$$\hat{OCB} = 90^\circ \text{ (radius } OC \text{ on tangent } BD\text{)}$$

$$\hat{CBO} = 180^\circ - 2x \text{ (sum of } \angle\text{'s in } \triangle BFC\text{)}$$

Step 3 : Then we move on to the other one.

In $\triangle OCD$

$$\hat{OCD} = 90^\circ \text{ (radius } OC \text{ on tangent } BD\text{)}$$

$$\hat{COD} = 180^\circ - 2x \text{ (sum of } \angle\text{'s in } \triangle OCE\text{)}$$

Step 4 : Again we have a common element.

Lastly, OC is a common side to both \triangle 's.

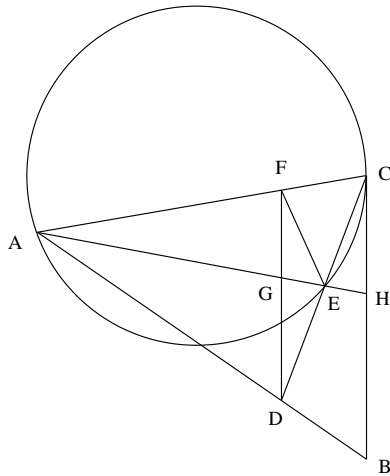
Step 5 : Then, once we've shown similarity, we use the proportionality relation, as well as our first observation, appropriately.

$$\begin{aligned} \therefore \triangle BOC & \parallel \triangle ODC \text{ (common side and 2 equal } \angle \text{'s)} \\ \therefore \frac{CO}{BC} &= \frac{CD}{CO} \\ \therefore \frac{OE}{BC} &= \frac{CD}{CO} \text{ (} OE = CD \text{ isosceles } \triangle OEC) \end{aligned}$$



Worked Example 191: Circle Geometry II

Question:



FD is drawn parallel to the tangent CB

Prove that:

1. $FADE$ is cyclic
2. $\triangle AFE \parallel \triangle CBD$
3. $\frac{FC \cdot AG}{GH} = \frac{DC \cdot FE}{BD}$

Answer

1. **Step 1 :** In this case, the best way to show $FADE$ is a cyclic quadrilateral is to look for equal angles, subtended by the same chord.

Let $\angle BCD = x$

$$\begin{aligned} \therefore \angle CAH &= x \text{ (} \angle \text{ between tangent } BC \text{ and chord } CE) \\ \therefore \angle FDC &= x \text{ (alternate } \angle \text{, } FD \parallel CB) \\ \therefore FADE &\text{ is a cyclic quadrilateral (chord } FE \text{ subtends equal } \angle \text{'s)} \end{aligned}$$

2. **Step 1 :** To show these 2 triangles similar we will need 3 equal angles. We can use the result from the previous question.

Let $\angle FEA = y$

$$\begin{aligned} \therefore \angle FDA &= y \text{ (} \angle \text{'s subtended by same chord } AF \text{ in cyclic quadrilateral } FADE) \\ \therefore \angle CBD &= y \text{ (corresponding } \angle \text{'s, } FD \parallel CB) \\ \therefore \angle FEA &= \angle CBD \end{aligned}$$

Step 2 : We have already proved 1 pair of angles equal in the previous question.

$$\angle BCD = \angle FAE \text{ (above)}$$

Step 3 : Proving the last set of angles equal is simply a matter of adding up the angles in the triangles. Then we have proved similarity.

$$\begin{aligned}\angle AFE &= 180^\circ - x - y \text{ (\(\angle\)'s in \(\triangle AFE\))} \\ \angle CBD &= 180^\circ - x - y \text{ (\(\angle\)'s in \(\triangle CBD\))} \\ \therefore \triangle AFE &\parallel\parallel \triangle CBD \text{ (3 \(\angle\)'s equal)}\end{aligned}$$

3. **Step 1 : This equation looks like it has to do with proportionality relation of similar triangles. We already showed triangles AFE and CBD similar in the previous question. So lets start there.**

$$\begin{aligned}\frac{DC}{BD} &= \frac{FA}{FE} \\ \therefore \frac{DC \cdot FE}{BD} &= FA\end{aligned}$$

Step 2 : Now we need to look for a hint about side FA . Looking at triangle CAH we see that there is a line FG intersecting it parallel to base CH . This gives us another proportionality relation.

$$\begin{aligned}\frac{AG}{GH} &= \frac{FA}{FC} \text{ (} FG \parallel CH \text{ splits up lines } AH \text{ and } AC \text{ proportionally)} \\ \therefore FA &= \frac{FC \cdot AG}{GH}\end{aligned}$$

Step 3 : We have 2 expressions for the side FA .

$$\therefore \frac{FC \cdot AG}{GH} = \frac{DC \cdot FE}{BD}$$

42.3 Co-ordinate Geometry

42.3.1 Equation of a Circle

We know that every point on the circumference of a circle is the same distance away from the centre of the circle. Consider a point (x_1, y_1) on the circumference of a circle of radius r with centre at (x_0, y_0) .

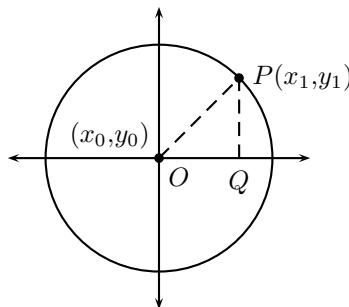


Figure 42.3: Circle h with centre (x_0, y_0) has a tangent, g passing through point P at (x_1, y_1) . Line f passes through the centre and point P .

In Figure 42.3, $\triangle OPQ$ is a right-angled triangle. Therefore, from the Theorem of Pythagoras, we know that:

$$OP^2 = PQ^2 + OQ^2$$

But,

$$\begin{aligned} PQ &= y_1 - y_0 \\ OQ &= x_1 - x_0 \\ OP &= r \\ \therefore r^2 &= (y_1 - y_0)^2 + (x_1 - x_0)^2 \end{aligned}$$

But, this same relation holds for any point P on the circumference. In fact, the relation holds for all points P on the circumference. Therefore, we can write:

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \quad (42.3)$$

for a circle with centre at (x_0, y_0) and radius r .

For example, the equation of a circle with centre $(0,0)$ and radius 4 is:

$$\begin{aligned} (y - y_0)^2 + (x - x_0)^2 &= r^2 \\ (y - 0)^2 + (x - 0)^2 &= 4^2 \\ y^2 + x^2 &= 16 \end{aligned}$$



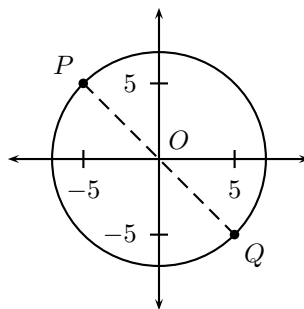
Worked Example 192: Equation of a Circle I

Question: Find the equation of a circle (centre O) with a diameter between two points, P at $(-5,5)$ and Q at $(5, -5)$.

Answer

Step 1 : Draw a picture

Draw a picture of the situation to help you figure out what needs to be done.



Step 2 : Find the centre of the circle

We know that the centre of a circle lies on the midpoint of a diameter. Therefore the co-ordinates of the centre of the circle is found by finding the midpoint of the line between P and Q . Let the co-ordinates of the centre of the circle be (x_0, y_0) , let the co-ordinates of P be (x_1, y_1) and let the co-ordinates of Q be (x_2, y_2) . Then,

the co-ordinates of the midpoint are:

$$\begin{aligned}x_0 &= \frac{x_1 + x_2}{2} \\ &= \frac{-5 + 5}{2} \\ &= 0 \\ y_0 &= \frac{y_1 + y_2}{2} \\ &= \frac{5 + (-5)}{2} \\ &= 0\end{aligned}$$

The centre point of line PQ and therefore the centre of the circle is at $(0,0)$.

Step 3 : Find the radius of the circle

If P and Q are two points on a diameter, then the radius is half the distance between them.

The distance between the two points is:

$$\begin{aligned}r = \frac{1}{2}PQ &= \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \frac{1}{2}\sqrt{(5 - (-5))^2 + (-5 - 5)^2} \\ &= \frac{1}{2}\sqrt{(10)^2 + (-10)^2} \\ &= \frac{1}{2}\sqrt{100 + 100} \\ &= \sqrt{\frac{200}{4}} \\ &= \sqrt{50}\end{aligned}$$

Step 4 : Write the equation of the circle

$$x^2 + y^2 = 50$$



Worked Example 193: Equation of a Circle II

Question: Find the center and radius of the circle

$$x^2 - 14x + y^2 + 4y = -28.$$

Answer

Step 1 : Change to standard form

We need to rewrite the equation in the form $(x - x_0)^2 + (y - y_0)^2 = r^2$

To do this we need to complete the square

i.e. add and subtract $(\frac{1}{2}$ coefficient of x)² and $(\frac{1}{2}$ coefficient of y)²

Step 2 : Adding coefficients

$$x^2 - 14x + y^2 + 4y = -28$$

$$\therefore x^2 - 14x + (7)^2 - (7)^2 + y^2 + 4y + (2)^2 - (2)^2 = -28$$

Step 3 : Complete the squares

$$\therefore (x - 7)^2 - (7)^2 + (y + 2)^2 - (2)^2 = -28$$

Step 4 : Take the constants to the other side

$$\therefore (x - 7)^2 - 49 + (y + 2)^2 - 4 = -28$$

$$\therefore (x - 7)^2 + (y + 2)^2 = -28 + 49 + 4$$

$$\therefore (x - 7)^2 + (y + 2)^2 = 25$$

Step 5 : Read the values from the equation

\therefore center is $(7; -2)$ and the radius is 5 units

42.3.2 Equation of a Tangent to a Circle at a Point on the Circle

We are given that a tangent to a circle is drawn through a point P with co-ordinates (x_1, y_1) . In this section, we find out how to determine the equation of that tangent.

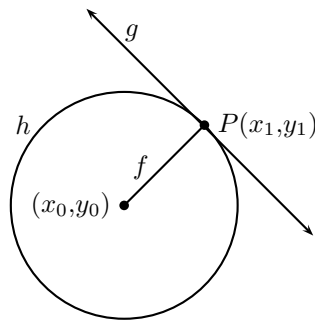


Figure 42.4: Circle h with centre (x_0, y_0) has a tangent, g passing through point P at (x_1, y_1) . Line f passes through the centre and point P .

We start by making a list of what we know:

1. We know that the equation of the circle with centre (x_0, y_0) is $(x - x_0)^2 + (y - y_0)^2 = r^2$.
2. We know that a tangent is perpendicular to the radius, drawn at the point of contact with the circle.

As we have seen in earlier grades, there are two steps to determining the equation of a straight line:

Step 1: Calculate the gradient of the line, m .

Step 2: Calculate the y -intercept of the line, c .

The same method is used to determine the equation of the tangent. First we need to find the gradient of the tangent. We do this by finding the gradient of the line that passes through the centre of the circle and point P (line f in Figure 42.4), because this line is a radius line and the tangent is perpendicular to it.

$$m_f = \frac{y_1 - y_0}{x_1 - x_0} \quad (42.4)$$

The tangent (line g) is perpendicular to this line. Therefore,

$$m_f \times m_g = -1$$

So,

$$m_g = -\frac{1}{m_f}$$

Now, we know that the tangent passes through (x_1, y_1) so the equation is given by:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - y_1 &= -\frac{1}{m_f}(x - x_1) \\ y - y_1 &= -\frac{1}{\frac{y_1 - y_0}{x_1 - x_0}}(x - x_1) \\ y - y_1 &= -\frac{x_1 - x_0}{y_1 - y_0}(x - x_1) \end{aligned}$$

For example, find the equation of the tangent to the circle at point $(1, 1)$. The centre of the circle is at $(0, 0)$. The equation of the circle is $x^2 + y^2 = 2$.

Use

$$y - y_1 = -\frac{x_1 - x_0}{y_1 - y_0}(x - x_1)$$

with $(x_0, y_0) = (0, 0)$ and $(x_1, y_1) = (1, 1)$.

$$\begin{aligned} y - y_1 &= -\frac{x_1 - x_0}{y_1 - y_0}(x - x_1) \\ y - 1 &= -\frac{1 - 0}{1 - 0}(x - 1) \\ y - 1 &= -\frac{1}{1}(x - 1) \\ y &= -(x - 1) + 1 \\ y &= -x + 1 + 1 \\ y &= -x + 2 \end{aligned}$$



Exercise: Co-ordinate Geometry

- Find the equation of the circle:
 - with center $(0; 5)$ and radius 5
 - with center $(2; 0)$ and radius 4
 - with center $(5; 7)$ and radius 18
 - with center $(-2; 0)$ and radius 6
 - with center $(-5; -3)$ and radius $\sqrt{3}$
- Find the equation of the circle with center $(2; 1)$ which passes through $(4; 1)$.
 - Where does it cut the line $y = x + 1$?
 - Draw a sketch to illustrate your answers.
- Find the equation of the circle with center $(-3; -2)$ which passes through $(1; -4)$.
 - Find the equation of the circle with center $(3; 1)$ which passes through $(2; 5)$.
 - Find the point where these two circles cut each other.
- Find the center and radius of the following circles:
 - $(x - 9)^2 + (y - 6)^2 = 36$
 - $(x - 2)^2 + (y - 9)^2 = 1$
 - $(x + 5)^2 + (y + 7)^2 = 12$
 - $(x + 4)^2 + (y + 4)^2 = 23$
 - $3(x - 2)^2 + 3(y + 3)^2 = 12$

$$F \quad x^2 - 3x + 9 = y^2 + 5y + 25 = 17$$

5. Find the x - and y - intercepts of the following graphs and draw a sketch to illustrate your answer:

A $(x + 7)^2 + (y - 2)^2 = 8$

B $x^2 + (y - 6)^2 = 100$

C $(x + 4)^2 + y^2 = 16$

D $(x - 5)^2 + (y + 1)^2 = 25$

6. Find the center and radius of the following circles:

A $x^2 + 6x + y^2 - 12y = -20$

B $x^2 + 4x + y^2 - 8y = 0$

C $x^2 + y^2 + 8y = 7$

D $x^2 - 6x + y^2 = 16$

E $x^2 - 5x + y^2 + 3y = -\frac{3}{4}$

F $x^2 - 6nx + y^2 + 10ny = 9n^2$

7. Find the equations to the tangent to the circle:

A $x^2 + y^2 = 17$ at the point $(1; 4)$

B $x^2 + y^2 = 25$ at the point $(3; 4)$

C $(x + 1)^2 + (y - 2)^2 = 25$ at the point $(3; 5)$

D $(x - 2)^2 + (y - 1)^2 = 13$ at the point $(5; 3)$

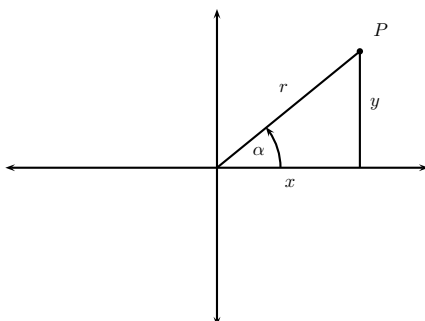
42.4 Transformations

42.4.1 Rotation of a Point about an angle θ

First we will find a formula for the co-ordinates of P after a rotation of θ .

We need to know something about polar co-ordinates and compound angles before we start.

Polar co-ordinates



Notice that $\sin \alpha = \frac{y}{r} \therefore y = r \sin \alpha$
 and $\cos \alpha = \frac{x}{r} \therefore x = r \cos \alpha$
 so P can be expressed in two ways:

$P(x; y)$ rectangular co-ordinates

or $P(r \cos \alpha; r \sin \alpha)$ polar co-ordinates.

Compound angles

(See derivation of formulae in Ch. 12)

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

Now consider P' after a rotation of θ

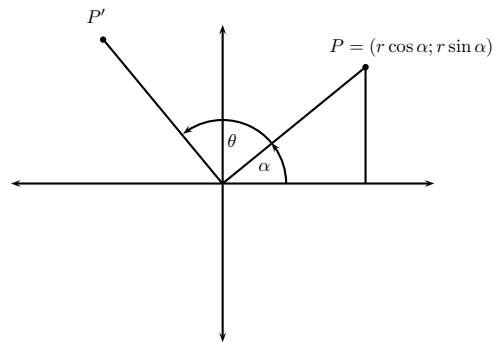
$$P(x; y) = P(r \cos \alpha; r \sin \alpha)$$

$$P'(r \cos(\alpha + \theta); r \sin(\alpha + \theta))$$

Expand the co-ordinates of P'

$$\begin{aligned} x - \text{co-ordinate of } P' &= r \cos(\alpha + \theta) \\ &= r [\cos \alpha \cos \theta - \sin \alpha \sin \theta] \\ &= r \cos \alpha \cos \theta - r \sin \alpha \sin \theta \\ &= x \cos \theta - y \sin \theta \end{aligned}$$

$$\begin{aligned} y - \text{co-ordinate of } P' &= r \sin(\alpha + \theta) \\ &= r [\sin \alpha \cos \theta + \sin \theta \cos \alpha] \\ &= r \sin \alpha \cos \theta + r \cos \alpha \sin \theta \\ &= y \cos \theta + x \sin \theta \end{aligned}$$



which gives the formula $P' = [(x \cos \theta - y \sin \theta); (y \cos \theta + x \sin \theta)]$.

So to find the co-ordinates of $P(1; \sqrt{3})$ after a rotation of 45° , we arrive at:

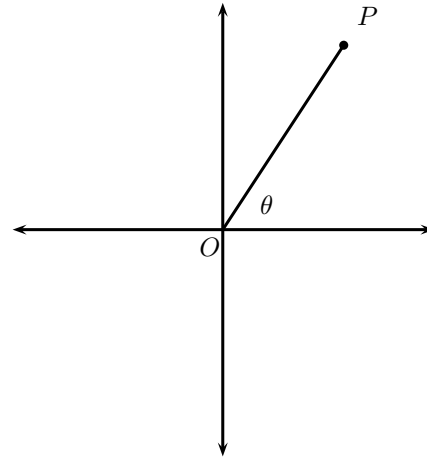
$$\begin{aligned} P' &= [(x \cos \theta - y \sin \theta); (y \cos \theta + x \sin \theta)] \\ &= [(1 \cos 45^\circ - \sqrt{3} \sin 45^\circ); (\sqrt{3} \cos 45^\circ + 1 \sin 45^\circ)] \\ &= \left[\left(\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{\sqrt{2}} \right); \left(\frac{\sqrt{3}}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) \right] \\ &= \left(\frac{1 - \sqrt{3}}{\sqrt{2}}; \frac{\sqrt{3} + 1}{\sqrt{2}} \right) \end{aligned}$$



Exercise: Rotations

Any line OP is drawn (not necessarily in the first quadrant), making an angle of θ degrees with the x -axis. Using the co-ordinates of P and the angle α , calculate the co-ordinates of P' , if the line OP is rotated about the origin through α degrees.

	P	α
1.	(2, 6)	60°
2.	(4, 2)	30°
3.	(5, -1)	45°
4.	(-3, 2)	120°
5.	(-4, -1)	225°
6.	(2, 5)	-150°



42.4.2 Characteristics of Transformations

Rigid transformations like translations, reflections, rotations and glide reflections preserve shape and size, and that enlargement preserves shape but not size.

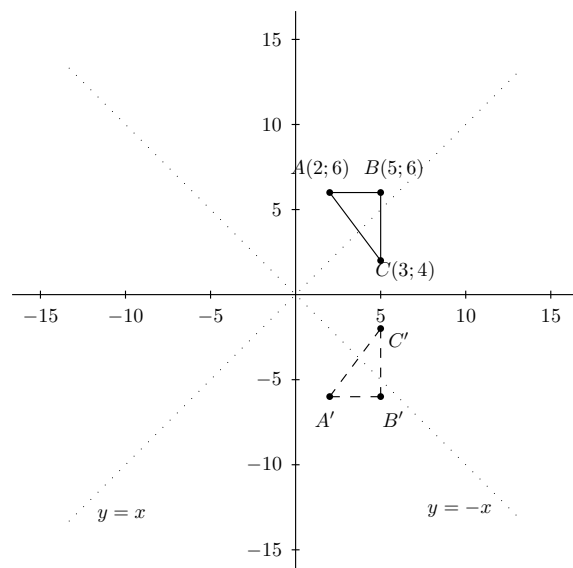
42.4.3 Characteristics of Transformations

Rigid transformations like translations, reflections, rotations and glide reflections preserve shape and size, and that enlargement preserves shape but not size.

Activity :: : Geometric Transformations

Draw a large 15×15 grid and plot $\triangle ABC$ with $A(2; 6)$, $B(5; 6)$ and $C(5; 1)$. Fill in the lines $y = x$ and $y = -x$.

Complete the table below, by drawing the images of $\triangle ABC$ under the given transformations. The first one has been done for you.



Transformation	Description (translation, reflection, rotation, enlargement)	Co-ordinates	Lengths	Angles
$(x; y) \rightarrow (x; -y)$	reflection about the x -axis	$A'(2; -6)$ $B'(5; -6)$ $C'(5; -2)$	$A'B' = 3$ $B'C' = 4$ $A'C' = 5$	$\hat{B}' = 90^\circ$ $\tan \hat{A} = 4/3$ $\therefore \hat{A} = 53^\circ, \hat{C} = 37^\circ$
$(x; y) \rightarrow (x + 1; y - 2)$				
$(x; y) \rightarrow (-x; y)$				
$(x; y) \rightarrow (-y; x)$				
$(x; y) \rightarrow (-x; -y)$				
$(x; y) \rightarrow (2x; 2y)$				
$(x; y) \rightarrow (y; x)$				
$(x; y) \rightarrow (y; x + 1)$				

A transformation that leaves lengths and angles unchanged is called a rigid transformation. Which of the above transformations are rigid?

42.5 Exercises

1. $\triangle ABC$ undergoes several transformations forming $\triangle A'B'C'$. Describe the relationship between the angles and sides of $\triangle KLM$ and $\triangle A'B'C'$ (e.g., they are twice as large, the same, etc.)

Transformation	Sides	Angles	Area
Reflect			
Reduce by a scale factor of 3			
Rotate by 90°			
Translate 4 units right			
Enlarge by a scale factor of 2			

2. $\triangle DEF$ has $\hat{E} = 30^\circ$, $DE = 4$ cm, $EF = 5$ cm. $\triangle DEF$ is enlarged by a scale factor of 6 to form $\triangle D'E'F'$.
- A Solve $\triangle DEF$
- B Hence, solve $\triangle D'E'F'$
3. $\triangle XYZ$ has an area of 6 cm^2 . Find the area of $\triangle X'Y'Z'$ if the points have been transformed as follows:
- A $(x, y) \rightarrow (x + 2; y + 3)$
- B $(x, y) \rightarrow (y; x)$

C $(x, y) \rightarrow (4x; y)$

D $(x, y) \rightarrow (3x; y + 2)$

E $(x, y) \rightarrow (-x; -y)$

F $(x, y) \rightarrow (x; -y + 3)$

G $(x, y) \rightarrow (4x; 4y)$

H $(x, y) \rightarrow (-3x; 4y)$

Appendix A

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